Automated Reasoning

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Natural Deduction in Propositional Logic

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Logic Puzzles

1. Tomorrow will be sunny or rainy.
   Tomorrow will not be sunny.
   What will the weather be tomorrow?
2. I like classical or pop music.
   If I like classical music, then I am sophisticated.
   I don't like pop music.
   Am I sophisticated?
3. Fred bought milk or Fred bought lemonade.
   Fred bought milk or Fred bought water.
   Fred did not buy both water and lemonade.
   What did Fred buy?

Syntax of Propositional Logic

Propositional logic represents the problems we have just seen by using **symbols** to represent (atomic) **propositions**.
These can be combined using the following **connectives**:

Name	symbol	usage
not	$\neg$	$\neg\phi$
and	$\wedge$	$\phi \wedge \psi$
or	$\vee$	$\phi \vee \psi$
implies	$\longrightarrow$	$\phi \longrightarrow \psi$
if and only if	$\longleftrightarrow$	$\phi \longleftrightarrow \psi$

precedence $\longrightarrow$

Assume all binary connectives right associative (Isabelle)

Example

1. $(SunnyTomorrow \vee RainyTomorrow) \wedge (\neg SunnyTomorrow)$
2. $(Class \vee Pop) \wedge (Class \longrightarrow Soph) \wedge \neg Pop$
3. $(M \vee L) \wedge (M \vee W) \wedge \neg(L \wedge W)$

Syntax (II)

The meaning of some statements can be ambiguous:

$$Class \vee Pop \wedge Class \longrightarrow Soph \longrightarrow \neg Pop.$$

We can use brackets to disambiguate a statement:

$$(Class \vee Pop) \wedge (Class \longrightarrow (Soph \longrightarrow (\neg Pop)))\,.$$

However, some brackets can be removed since the operators have a precedence and associativity:

$$(Class \vee Pop) \wedge (Class \longrightarrow Soph \longrightarrow \neg Pop)\,.$$

Note that

$$A \vee B \wedge C \quad \text{denotes} \quad A \vee (B \wedge C).$$

Also note that implication is right associative, so

$$P \longrightarrow Q \longrightarrow R \quad \text{denotes} \quad P \longrightarrow (Q \longrightarrow R).$$

Formal Syntax

A syntactically correct formula is called a **well-formed formula (wff)**.
Given a (possibly infinite) alphabet of propositional symbols $\mathcal{L}$, the set of wffs is the smallest set such that

- any symbol $P \in \mathcal{L}$ is a wff;
- if ϕ and ψ are wffs, so are $\neg\phi$, $\phi \vee \psi$, $\phi \wedge \psi$, $\phi \longrightarrow \psi$, $\phi \longleftrightarrow \psi$;
- if ϕ is a wff, then (ϕ) is a wff.

When interested in *abstract* syntax (tree-structure of formulas) rather than *concrete* syntax, we forget last clause.

Semantics

Each wff is assigned a meaning or **semantics**, **T** or **F**, depending on whether its constituent wffs are assigned **T** or **F**.
Truth tables are one way to assign truth values to wffs.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	$\neg P$
T	F
F	T

P	Q	$P \longrightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Semantics of Weather Problem

① Tomorrow will be sunny or rainy.

② Tomorrow will not be sunny.

What will the weather be tomorrow?

$SunnyTomorrow$	$RainyTomorrow$	① $S \vee R$	② $\neg S$	① $\wedge$ ② $(S \vee R) \wedge \neg S$
T	**T**	**T**	**F**	**F**
T	**F**	**T**	**F**	**F**
F	**T**	**T**	**T**	**T**
F	**F**	**F**	**T**	**F**

The highlighted row is the only possibility satisfying the two constraints in our weather problem.
We can see from this that it will rain tomorrow.

Exercise: Fred's Drink Problem

1. ① Fred bought milk or Fred bought lemonade.
2. ② Fred bought milk or Fred bought water.
3. ③ Fred did not buy both water and lemonade.

What did Fred buy?

M	L	W	① $M \vee L$	② $M \vee W$	③ $\neg(L \wedge W)$	① $\wedge$ ② $\wedge$ ③
T	**T**	**T**				
T	**T**	**F**				
T	**F**	**T**				
T	**F**	**F**				
F	**T**	**T**				
F	**T**	**F**				
F	**F**	**T**				
F	**F**	**F**				

Answer: Fred's Drink Problem

1. ① Fred bought milk or Fred bought lemonade.
2. ② Fred bought milk or Fred bought water.
3. ③ Fred did not buy both water and lemonade.

What did Fred buy?
Fred bought either:

- milk and lemonade
- milk and water
- milk

M	L	W	① $M \vee L$	② $M \vee W$	③ $\neg(L \wedge W)$	① $\wedge$ ② $\wedge$ ③
T	T	T	T	T	F	F
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	T	T	F	F
F	T	F	T	F	T	F
F	F	T	F	T	T	F
F	F	F	F	F	T	F

Truth tables are a **complete method** and can easily be automated. *But*, a wff with n symbols needs a table with 2^n rows. This is exponential in n, so impractical for large values of n.

Semantics: Important Definitions

Definition (Interpretation)

An interpretation is a **truth assignment** to the symbols in the alphabet $\mathcal{L}$: it is a function from $\mathcal{L}$ to $\{\mathbf{T}, \mathbf{F}\}$.

Example

$\mathit{SunnyTomorrow} \mapsto \mathbf{F}$ and $\mathit{RainyTomorrow} \mapsto \mathbf{T}$

Using truth-tables, an interpretation naturally extends to all formulas built using propositional symbols from $\mathcal{L}$.

Definition (Satisfaction)

An interpretation **satisfies** a wff if it makes it have value $\mathbf{T}$.

Definition (Satisfiable)

A wff is satisfiable if there is **some interpretation** which satisfies it.
A wff is unsatisfiable if it is not satisfiable.

Definition (Valid or tautology)

A wff is valid or a tautology if **every interpretation satisfies** it.

Example

Is $P \vee Q$ satisfiable, unsatisfiable or a tautology? How about $P \wedge \neg P$ and $P \vee \neg P$?

Semantics: Important Definitions (II)

Definition (Entailment)

The wffs $\phi_1, \phi_2, \ldots, \phi_n$ *entail* ψ if, for any interpretation which satisfies all of $\phi_1, \phi_2, \ldots, \phi_n$ also satisfies the wff ψ. We then write $\phi_1, \phi_2, \ldots, \phi_n \models \psi$.
We can use truth-table analysis to identify correct entailments.

Note If there is *no* interpretation which satisfies all of $\phi_1, \phi_2, \ldots, \phi_n$, then $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ holds for any ψ.
Contradictions entail everything!
Note Everything entails a tautology. If ψ is a tautology, then $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ holds for any $\phi_1 \ldots \phi_n$. We then write $\models \psi$ to say that ψ is a tautology

Example

Is $\neg P, Q \models Q \wedge (P \rightarrow R)$ a valid entailment?

Inference rules

Inference rule

An inference rule tells us how one wff can be **derived** in 1 step from zero, one or more other wffs. We write

$$\frac{\phi_1 \quad \phi_2 \quad \dots \quad \phi_n}{\psi}\ R$$

if wff ψ is derived from wffs $\phi_1, \phi_2, \dots, \phi_n$ using rule R,

An example rule is **Conjunction Introduction**.

$$\frac{P \quad Q}{P \wedge Q}\ conjI$$

Strictly speaking, the P and Q here are **meta-variables**. This **rule schema** characterises an infinite number of **rule instances**, gotten by substituting wffs for the P and Q.

Validity of rules

- Inference rules must be **valid**. They must preserve truth.
- More formally, for all instances

$$\frac{\phi_1 \quad \phi_2 \quad \dots \quad \phi_n}{\psi}\ R$$

  of rule R we must have $\phi_1, \phi_2, \dots, \phi_n \models \psi$.
- Inference is **transitive**. If we can infer χ from ψ and we can infer ψ from ϕ, then we can infer χ from ϕ. This means we can chain deductions together to form a deduction "tree".

Formal Deductive system

- A formal deductive system is one which uses a **valid set of inference rules**.
- We will be looking at natural deduction developed by Gentzen and Prawitz.
- For every connective $*$, we have two kinds of inference rule:

Introduction

how can I derive $A * B$?

Elimination

what can I derive **from** $A * B$?

Ways of applying rules

Inference rules are applied in two basic ways.

Forward proof: if we derive new wffs from existing wffs by applying rules top down.

Backward proof: if we conjecture some wff true and apply rules bottom-up to produce new wffs from which the original wff is derived.

A Simple Proof

Assuming A and B, prove $A \wedge (B \wedge A)$

$$\dfrac{A \quad \dfrac{B \quad A}{B \wedge A}\ \mathit{conjI}}{A \wedge (B \wedge A)}\ \mathit{conjI}$$

$$\boxed{\dfrac{P \quad Q}{P \wedge Q}\ \mathit{conjI}}$$

Assumptions: $A \quad B$

Summary

- A first look at propositional logic.
  - Syntax
  - Semantics
- Natural deduction
  - introduction and elimination rules;
  - proofs given as trees
- Next time
  - more introduction and elimination rules
  - the rules of the game in Isabelle