Formalising the Foundations of Geometry

Phil Scott

March 27, 2015

Formalising Mathematics

Why formalise?

- Because some proofs are too hard to verify by inspection (Kepler conjecture, Four-Colour Theorem, ABC conjecture)
- We need to contribute *groundwork* for such projects.
- To investigate new *representations*.
- To investigate ways to *organise* mathematics.
- To add *case-studies*, pushing our theorem provers.
- To investigate new *automation* for new domains.
- For historical insight into pre $20^{th}/21^{st}$ century mathematics.

The Foundations of Geometry

Euclid's *Elements*

- Earliest extant text on axiomatic geometry.
- "possibly the most influential mathematical text ever written"
- A system of ruler and compass constructions.

The Foundations of Geometry

Euclid's *Elements*

- Earliest extant text on axiomatic geometry.
- "possibly the most influential mathematical text ever written"
- A system of ruler and compass constructions. But, enables number theory, algebra, the theory of proportion, solid geometry and integration proofs to compute areas and volumes.

Euclid's Axioms

`http://aleph0.clarku.edu/~djoyce/java/elements/elements.html`

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and radius.
4. That all right angles are equal to each other.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Axiomatic *Foundations of Geometry*

Hilbert's *Foundations of Geometry*

- "most influential book on geometry in a hundred years"
- 10 German editions. 2 English translations (last 1971).
- Truly formal: "Beer mugs, tables and chairs".
- Now 22 axioms covering *incidence of points of lines*, *ordering of points on a line*, segments and angles defined as point pairs and intersecting lines and their *congruence*, *a parallel axiom*, *the Archimedean axiom and a completeness axiom*.

Formalisation and Machine-Verification

In reviews:

Formalisation and Machine-Verification

In reviews:

- "Theoretically, at least, the deductions could be made without any reference to their content by the use of the ratiocinative calculus like that of Peano ($\exists$, $\vee$, $\supset$, $\sim$) or a Jevons Logic machine." — Veblen

Formalisation and Machine-Verification

In reviews:

- "Theoretically, at least, the deductions could be made without any reference to their content by the use of the ratiocinative calculus like that of Peano ($\exists$, $\vee$, $\supset$, $\sim$) or a Jevons Logic machine." — Veblen

- "We might put the axioms into a reasoning apparatus like the logical machine of Stanley Jevons, and see all geometry come out of it." — Poincaré

Formalisation and Machine-Verification

"This notion may seem artificial and peurile; and it is needless to point out how disastrous it would be in teaching and how hurtful to mental development; how deadening it would be for investigators, whose originality it would nip in the bud."

Formalisation and Machine-Verification

"I see in logistic only shackles for the inventor. It is no aid in conciseness — far from it, and if twenty-seven equations were necessary to establish that 1 is a number, how many would be needed to prove a real theorem?""

Hilbert's Primitives

"Consider three distinct sets of objects. Let the objects of the first set be called *points* [. . .]; let the objects of the second set be called *lines*"; let the objects of the third set be called *planes*.

. . .

The points, lines and planes are considered to have certain mutual relations and these relations are denoted by words like **"lie,"** **"between"**, [...] The precise and mathematically complete description of these relations follows from the **"axioms of geometry"**

[1] For simplicity, we'll just formalise the planar fragment.

Hilbert's Primitives

"Consider three distinct sets of objects. Let the objects of the first set be called *points* [. . .]; let the objects of the second set be called *lines*"; let the objects of the third set be called *planes*.

. . .

The points, lines and planes are considered to have certain mutual relations and these relations are denoted by words like **"lie,"** **"between"**, [...] The precise and mathematically complete description of these relations follows from the **"axioms of geometry"**

```
new_type ("point",0)
new_type ("line",0)
```
[1]

[1] For simplicity, we'll just formalise the planar fragment.

Hilbert's Primitives

"Consider three distinct sets of objects. Let the objects of the first set be called *points* [. . .]; let the objects of the second set be called *lines*"; let the objects of the third set be called *planes*.

. . .

The points, lines and planes are considered to have certain mutual relations and these relations are denoted by words like **"lie,"** **"between"**, [...] The precise and mathematically complete description of these relations follows from the **"axioms of geometry"**

```
new_type ("point",0)
new_type ("line",0)
```
[1]

```
new_constant ("on_line", `:(point -> line -> bool)`)
new_constant ("between",
              `:(point -> point -> point -> bool)`)
```

[1] For simplicity, we'll just formalise the planar fragment.

Hilbert's axioms

I, 1 *For every two points A, B there exists a line a that contains each of the points A, B.*

I, 2 *For every two points A, B there exits [sic] no more than one line that contains each of the points A, B.*

I, 3 *There exist at least two points on a line. There exist at least three points that do not lie on a line.*

Hilbert's axioms

I, 1 *For every two points A, B there exists a line a that contains each of the points A, B.*

I, 2 *For every two points A, B there exits [sic] no more than one line that contains each of the points A, B.*

I, 3 *There exist at least two points on a line. There exist at least three points that do not lie on a line.*

$$\vdash A \neq B \longrightarrow \exists a.\ \texttt{on\_line}\ A\ a \wedge \texttt{on\_line}\ B\ a \tag{I, 1}$$

$$\begin{aligned}\vdash A \neq B \wedge \texttt{on\_line}\ A\ a \wedge \texttt{on\_line}\ B\ a \\ \wedge\, \texttt{on\_line}\ A\ b \wedge \texttt{on\_line}\ B\ b \\ \longrightarrow a = b\end{aligned} \tag{I, 2}$$

$$\vdash \exists A.\ \exists B.\ A \neq B \wedge \texttt{on\_line}\ A\ a \wedge \texttt{on\_line}\ B\ a \tag{I, 3.1}$$

$$\vdash \exists A.\ \exists B.\ \exists C.\ \neg(\exists a.\ \texttt{on\_line}\ A\ a \wedge \texttt{on\_line}\ B\ a \wedge \texttt{on\_line}\ C\ a) \tag{I, 3.2}$$

Hilbert's axioms

II, 1 *If a point B lies between a point A and a point C then the points A, B, C are three distinct points of a line, and B then also lies between C and A.*

II, 2 *For two points A and C, there always exists at least one point B on the line AC such that C lies between A and B.*

II, 3 *Of any three points on a line there exists no more than one that lies between the other two.*

II, 1 *If a point B lies between a point A and a point C then the points A, B, C are three distinct points of a line, and B then also lies between C and A.*

II, 2 *For two points A and C, there always exists at least one point B on the line AC such that C lies between A and B.*

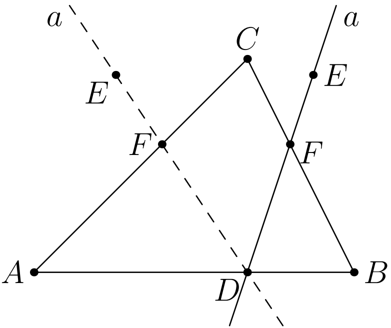
II, 3 *Of any three points on a line there exists no more than one that lies between the other two.*

$$\begin{aligned}\vdash &\texttt{between}\ A\ B\ C \longrightarrow A \neq C \\ &\wedge (\exists a.\ \texttt{on\_line}\ A\ a \wedge \texttt{on\_line}\ B\ a \wedge \texttt{on\_line}\ C\ a) \\ &\wedge \texttt{between}\ C\ B\ A\end{aligned} \qquad \text{(II, 1)}$$

$$\vdash A \neq B \longrightarrow \exists C.\ \texttt{between}\ A\ B\ C \qquad \text{(II, 2)}$$

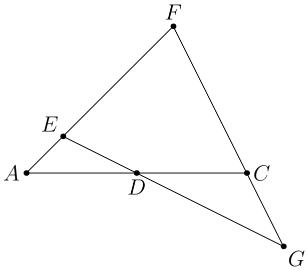
$$\vdash \texttt{between}\ A\ B\ C \longrightarrow \neg\texttt{between}\ A\ C\ B \qquad \text{(II, 3)}$$

II, 4 *Let A, B, C be three points that do not lie on a line and let a be a line in the plane ABC which does not meet any of the points A, B, C. If the line a passes through a point of the segment AB, it also passes through a point of the segment AC, or through a point of the segment BC.*



First Proof

THEOREM 3. For two points A and C there always exists at least one point D on the line AC that lies between A and C.
PROOF. By Axiom I, 3 there exists a point E outside the line AC and by Axiom II, 2 there exists on AE a point F such that E is a point of the segment AF. By the same axiom and by Axiom II, 3 there exists on FC a point G that does not lie on the segment FC. By Axiom II, 4 the line EG must then intersect the segment AC at a point D.



Representation

$$\begin{array}{rl} \texttt{collinear} & : (\texttt{point} \to \texttt{bool}) \to \texttt{bool} \\ \vdash_{def} \texttt{collinear Ps} & \iff \exists a.\, \forall P.\, P \in Ps \longrightarrow \texttt{on\_line}\ P\ a. \end{array}$$

Representation

$$\texttt{collinear} : (\texttt{point} \to \texttt{bool}) \to \texttt{bool}$$

$$\vdash_{def} \texttt{collinear Ps} \iff \exists a.\, \forall P.\, P \in Ps \longrightarrow \texttt{on\_line}\ P\ a.$$

$$\vdash \texttt{collinear}\ \{A, B\}$$

$$\vdash S \subseteq T \wedge \texttt{collinear}\ T \longrightarrow \texttt{collinear}\ S$$

$$\vdash A \neq B \wedge A, B \in S, T \longrightarrow \texttt{collinear}\ S \wedge \texttt{collinear}\ T$$

$$\longrightarrow \texttt{collinear}\ (S \cup T)$$

Representation

$$\texttt{collinear} : (\texttt{point} \to \texttt{bool}) \to \texttt{bool}$$
$$\vdash_{def} \texttt{collinear Ps} \iff \exists a.\, \forall P.\, P \in Ps \longrightarrow \texttt{on\_line}\ P\ a.$$

$$\vdash \texttt{collinear}\ \{A, B\}$$
$$\vdash S \subseteq T \wedge \texttt{collinear}\ T \longrightarrow \texttt{collinear}\ S$$
$$\vdash A \neq B \wedge A, B \in S, T \longrightarrow \texttt{collinear}\ S \wedge \texttt{collinear}\ T$$
$$\longrightarrow \texttt{collinear}\ (S \cup T)$$

$$\vdash \texttt{collinear}\ S \wedge \texttt{collinear}\ T \wedge \neg\texttt{collinear}\ U \wedge U \subseteq S \cup T$$
$$\wedge A, B \in S, T \longrightarrow A = B$$
$$\vdash \texttt{collinear}\ S \wedge \neg\texttt{collinear}\ \{A, B, C\}$$
$$\wedge X, Y, A, B \in S \wedge X \neq Y \longrightarrow \neg\texttt{collinear}\ \{C, X, Y\}$$

Pasch in collinear sets

$$
\begin{aligned}
&\vdash \neg\texttt{collinear}\ \{A, B, C\} \\
&\wedge \neg\texttt{collinear}\ \{A, D, E\} \\
&\wedge \neg\texttt{collinear}\ \{C, D, E\} \\
&\wedge \texttt{between}\ A\ D\ B \\
&\longrightarrow \exists F.\ \texttt{collinear}\ \{D, E, F\} \\
&\qquad \wedge (\texttt{between}\ A\ F\ C \vee \texttt{between}\ B\ F\ C).
\end{aligned}
\tag{II, 4}
$$

Verification of Theorem 3

$\texttt{assume } A \neq C$

$\texttt{so consider } E \texttt{ such that}$

$\quad \neg(\exists a.\ \texttt{on\_line } A\ a \wedge \texttt{on\_line } C\ a \wedge \texttt{on\_line } E\ a)$

$\quad \texttt{by } (\mathsf{I},\ 2), (\mathsf{I},\ 3.2)$ 0

$\texttt{obviously by\_neqs consider } F \texttt{ such that between } A\ E\ F$

$\quad \texttt{from } 0 \texttt{ by } (\mathsf{II},\ 2)$ 1

$\texttt{obviously by\_neqs so consider } G \texttt{ such that between } F\ C\ G$

$\quad \texttt{from } 0 \texttt{ by } (\mathsf{II},\ 2)$ 2

$\texttt{obviously by\_incidence so consider } D \texttt{ such that}$

$\quad (\exists a.\ \texttt{on\_line } E\ a \wedge \texttt{on\_line } G\ a \wedge \texttt{on\_line } D\ a)$

$\quad \wedge (\texttt{between } A\ D\ C \vee \texttt{between } F\ D\ C)$

$\quad \texttt{using K (MATCH\_MP\_TAC } (\mathsf{II},\ 4)) \texttt{ from } 0, 1$

$\texttt{obviously } (\texttt{by\_eqs} \circ \texttt{split}) \texttt{ qed from } 0, 1, 2 \texttt{ by } (\mathsf{II},\ 1), (\mathsf{II},\ 3)$

Ordering along a line

THEOREM 4. Of any three points A, B, C on a line there always is one that lies between the other two.

Ordering along a line

THEOREM 4. Of any three points A, B, C on a line there always is one that lies between the other two.

$$\begin{aligned}&\vdash \texttt{on\_line}\ A\ a \wedge \texttt{on\_line}\ B\ a \wedge \texttt{on\_line}\ C\ a \\ &\quad \wedge\ A \neq B \wedge A \neq C \wedge B \neq C \\ &\quad \longrightarrow \texttt{between}\ A\ B\ C \vee \texttt{between}\ B\ A\ C \vee \texttt{between}\ A\ C\ B\end{aligned} \tag{THEOREM 4}$$

THEOREM 5. Given any four points on a line, it is always possible to label them A, B, C, D in such a way that the point labelled B lies between A and C and also between A and D, and furthermore, that the point labelled C lies between A and D and also between B and D.

THEOREM 5. Given any four points on a line, it is always possible to label them A, B, C, D in such a way that the point labelled B lies between A and C and also between A and D, and furthermore, that the point labelled C lies between A and D and also between B and D.

$$\vdash \left(\begin{array}{l}\texttt{between}\ A\ B\ C \wedge \texttt{between}\ B\ C\ D \\ \qquad\quad \longrightarrow \texttt{between}\ A\ B\ D \wedge \texttt{between}\ A\ C\ D\end{array}\right)$$
$$\wedge \left(\begin{array}{l}\texttt{between}\ A\ B\ C \wedge \texttt{between}\ A\ C\ D \\ \qquad\quad \longrightarrow \texttt{between}\ A\ B\ D \wedge \texttt{between}\ B\ C\ D\end{array}\right) \tag{THEOREM 5}$$

Too many case splits

so consider R such that between $A\ R\ B$ 7

have between $P\ A\ Q \wedge$ between $P\ B\ Q$ from 6 by... 8

hence between $P\ R\ Q$ from $6, 7$ by ... [?]

Too many case splits

$\texttt{so consider}\ R\ \texttt{such that between}\ A\ R\ B$ 7

$\texttt{have between}\ P\ A\ Q \wedge \texttt{between}\ P\ B\ Q\ \texttt{from}\ 6\ \texttt{by}\ldots$ 8

$\texttt{hence between}\ P\ R\ Q\ \texttt{from}\ 6, 7\ \texttt{by}\ \ldots\ [?]$

- Where is B in relation to A and P?

Too many case splits

$\texttt{so consider } R \texttt{ such that between } A\ R\ B$ 7

$\texttt{have between } P\ A\ Q \wedge \texttt{between } P\ B\ Q \texttt{ from } 6 \texttt{ by} \ldots$ 8

$\texttt{hence between } P\ R\ Q \texttt{ from } 6,7 \texttt{ by } \ldots$ [?]

- Where is B in relation to A and P?
- If between A and P, then we can reason transitively that Q is between P and Q.

Too many case splits

`so consider` R `such that between` $A\ R\ B$ 7

`have between` $P\ A\ Q \wedge$ `between` $P\ B\ Q$ `from` 6 `by...` 8

`hence between` $P\ R\ Q$ `from` $6, 7$ `by ...` [?]

- Where is B in relation to A and P?
- If between A and P, then we can reason transitively that Q is between P and Q.
- We know P cannot be between A and B.

Too many case splits

`so consider` R `such that between` $A\ R\ B$ 7

`have between` $P\ A\ Q \wedge$ `between` $P\ B\ Q$ `from` 6 `by` $\ldots$ 8

`hence between` $P\ R\ Q$ `from` $6,7$ `by` $\ldots$ [?]

- Where is B in relation to A and P?
- If between A and P, then we can reason transitively that Q is between P and Q.
- We know P cannot be between A and B.
- If A is between P and B:

Too many case splits

`so consider` R `such that between` $A\ R\ B$ 7

`have between` $P\ A\ Q \wedge$ `between` $P\ B\ Q$ `from` 6 `by`... 8

`hence between` $P\ R\ Q$ `from` $6,7$ `by` ... [?]

- Where is B in relation to A and P?
- If between A and P, then we can reason transitively that Q is between P and Q.
- We know P cannot be between A and B.
- If A is between P and B:
  - Where is B in relation to A and Q?

Too many case splits

so consider R such that between A R B 7

have between P A $Q \wedge$ between P B Q from 6 by... 8

hence between P R Q from 6, 7 by ... [?]

- Where is B in relation to A and P?
- If between A and P, then we can reason transitively that Q is between P and Q.
- We know P cannot be between A and B.
- If A is between P and B:
  - Where is B in relation to A and Q?
  - If between A and Q, we can again reason transitively.

Too many case splits

```
so consider R such that between A R B                    7
have between P A Q ∧ between P B Q from 6 by...          8
hence between P R Q from 6,7 by ... [?]
```

- Where is B in relation to A and P?
- If between A and P, then we can reason transitively that Q is between P and Q.
- We know P cannot be between A and B.
- If A is between P and B:
  - Where is B in relation to A and Q?
  - If between A and Q, we can again reason transitively.
  - If A is between B and Q, we reason transitively to a contradiction.

Too many case splits

$\texttt{so consider } R \texttt{ such that between } A\ R\ B$ 7
$\texttt{have between } P\ A\ Q \wedge \texttt{between } P\ B\ Q \texttt{ from } 6 \texttt{ by} \ldots$ 8
$\texttt{hence between } P\ R\ Q \texttt{ from } 6,7 \texttt{ by } \ldots$ [?]

- Where is B in relation to A and P?
- If between A and P, then we can reason transitively that Q is between P and Q.
- We know P cannot be between A and B.
- If A is between P and B:
  - Where is B in relation to A and Q?
  - If between A and Q, we can again reason transitively.
  - If A is between B and Q, we reason transitively to a contradiction.
  - If Q is between A and B, we again reason transitively to a contradiction.

Reduce to ordering of naturals

$$
\begin{aligned}
&\vdash \texttt{finite}\ X \land \texttt{collinear}\ X \\
&\quad \longrightarrow \exists f.\ \forall A.\ \forall B.\ \forall C.\ A \in X \land B \in X \land C \in X \\
&\qquad \longrightarrow \begin{pmatrix} \texttt{between}\ A\ B\ C \\ \qquad \iff (f\ A < f\ B \land f\ B < f\ C) \\ \qquad\qquad \lor (f\ C < f\ B \land f\ B < f\ A) \end{pmatrix} \\
&\qquad \land \forall A.\ \forall B.\ A \in X \land B \in X \longrightarrow (A = B \iff f\ A = f\ B).
\end{aligned}
$$

Reduce to ordering of naturals

$$\vdash \texttt{finite}\ X \wedge \texttt{collinear}\ X$$
$$\longrightarrow \exists f.\ \forall A.\ \forall B.\ \forall C.\ A \in X \wedge B \in X \wedge C \in X$$
$$\longrightarrow \left(\begin{array}{l}\texttt{between}\ A\ B\ C \\ \qquad \iff (f\ A < f\ B \wedge f\ B < f\ C) \\ \qquad\qquad\qquad \vee (f\ C < f\ B \wedge f\ B < f\ A)\end{array}\right)$$
$$\wedge\, \forall A.\ \forall B.\ A \in X \wedge B \in X \longrightarrow (A = B \iff f\ A = f\ B).$$

so consider R such that between $A\ R\ B$ 7

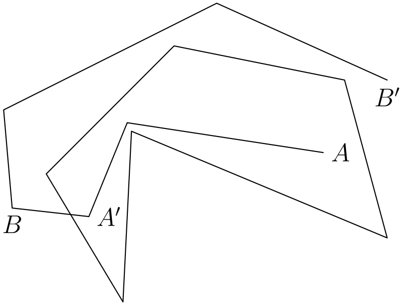
have between $P\ A\ Q \wedge$ between $P\ B\ Q$ from 6 by ... 8

hence between $P\ R\ Q$ from 6, 7

using ORDER_TAC $\{P, Q, R, A, B\}$

Jordan Curve Theorem for Polygons

THEOREM 9. Every single [simple] polygon lying in a plane α separates the points of the plane α that are not on the polygonal segment of the polygon into two regions, [. . .].



Formalisation of Polygonal Segments

$$\texttt{adjacent} \;:\; [] \rightarrow [(point, point)]$$

$$\begin{aligned}
&\texttt{adjacent}\,[P_0, P_1, P_2, \ldots, P_n] \\
&\quad = \texttt{zip}\,(\texttt{butlast}\,[P_0, P_1, P_2, \ldots, P_n])\,(\texttt{tail}\,[P_0, P_1, P_,2, \ldots, P_n]) \\
&\quad = \texttt{zip}\;\begin{array}{lccccc} [& P_0, & P_1, & P_2, & \ldots, & P_{n-1} &] \\ [& P_1, & P_2, & P_3, & \ldots, & P_n &] \end{array} \\
&\quad = [(P_0, P_1), (P_1, P_2), (P_2, P_3), \ldots, (P_{n-1}, P_n)]
\end{aligned}$$

$$\texttt{on\_polypath} \;:\; [\texttt{point}] \rightarrow \texttt{point} \rightarrow \texttt{bool}$$

$$\begin{aligned}
&\texttt{on_polypath}\; Ps\; P \\
&\quad \iff \texttt{mem}\; P\; Ps \lor \exists x\; y.\, \texttt{mem}\,(x, y)\,\texttt{adjacent}\; Ps \land \texttt{between}\; x\; P\; y
\end{aligned}$$

Simple polygons

$$
\begin{array}{l}
\texttt{simple\_polygon} : [\texttt{point}] \to \texttt{bool} \\
\vdash_{def} \texttt{simple\_polygon}\ Ps \iff \\
\quad 3 \leq \texttt{length}\ Ps \\
\quad \wedge\, \texttt{head}\ ps = \texttt{last}\ Ps \\
\quad \wedge\, \texttt{pairwise}\ (\neq)\ (\texttt{butlast}\ Ps) \\
\quad \wedge\, \neg(\exists P.\ \exists Q.\ \exists X. \\
\qquad \big(\texttt{mem}\ X\ Ps \wedge \texttt{mem}\ (P, Q)\ (\texttt{adjacent}\ Ps) \wedge \texttt{between}\ P\ X\ Q\ \ \big) \\
\quad \wedge\, \texttt{pairwise}\ (\lambda(P, Q)\ (P', Q'). \\
\qquad\quad \neg(\exists X.\ \texttt{between}\ P\ X\ P' \wedge \texttt{between}\ Q\ X\ Q')\ (\texttt{adjacent}\ Ps)).
\end{array}
$$

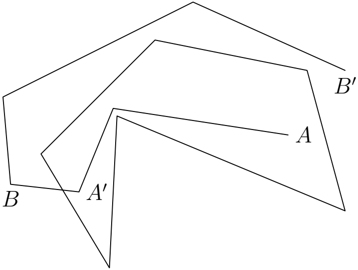
Polygonal Jordan Curve Theorem

$$\vdash \texttt{simple\_polygon}\ Ps \\ \longrightarrow \exists P.\ \exists Q.\ \neg\texttt{on\_polypath}\ Ps\ P \wedge \neg\texttt{on\_polypath}\ Ps\ Q \\ \wedge \neg\texttt{polypath\_connected}\ (\texttt{on\_polypath}\ Ps)\ P\ Q$$

$$\vdash \texttt{simple\_polygon}\ Ps \\ \wedge \neg\texttt{on\_polypath}\ Ps\ P \wedge \neg\texttt{on\_polypath}\ Ps\ Q \wedge \neg\texttt{on\_polypath}\ Ps\ R \\ \longrightarrow \texttt{polypath\_connected}\ (\texttt{on\_polypath}\ Ps)\ P\ Q \\ \vee \texttt{polypath\_connected}\ (\texttt{on\_polypath}\ Ps)\ P\ R \\ \vee \texttt{polypath\_connected}\ (\texttt{on\_polypath}\ Ps)\ Q\ R$$

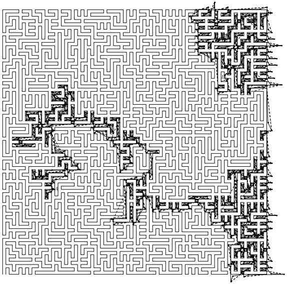
Jordan Curve Theorem for Polygons

"With the aid of Theorem 8, one obtains [the theorem] without much difficulty." — Hilbert



Jordan Curve Theorem for Polygons

"With the aid of Theorem 8, one obtains [the theorem] ~~without much difficulty~~." — Hilbert



Prose Proofs

- Almost all useless without more axioms.

Prose Proofs

- Almost all useless without more axioms.
- Plumb line proof requires that we can cast rays in a fixed direction.

Prose Proofs

- Almost all useless without more axioms.
- Plumb line proof requires that we can cast rays in a fixed direction.
- Winding proof assumes a theory of angles.

Prose Proofs

- Almost all useless without more axioms.
- Plumb line proof requires that we can cast rays in a fixed direction.
- Winding proof assumes a theory of angles.
- Both require reasoning about continuity.

Veblen to the Rescue?

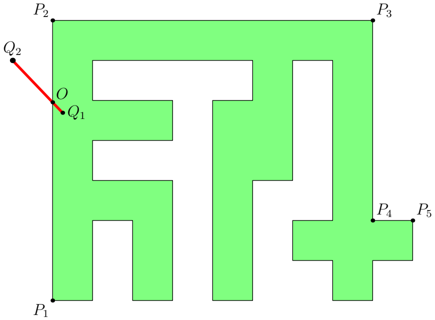
"[Jordan's] proof, however, is unsatisfactory to many mathematicians. It assumes the theorem without proof in the important special case of a simple polygon." — Veblen

Veblen to the Rescue?

"[Jordan's] proof, however, is unsatisfactory to many mathematicians. It assumes the theorem without proof in the important special case of a simple polygon." — Veblen

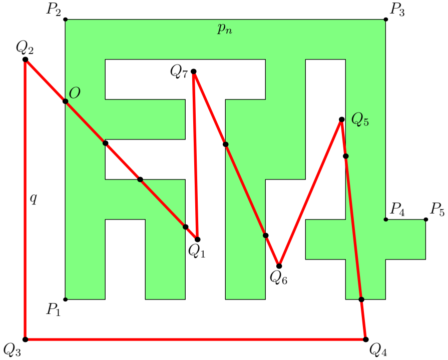
"[Veblen's] proof was part of his larger project to axiomatise analysis situs as an isolated field of mathematics. The model for this project was Hilbert's axiomatisation of the foundations of geometry in 1899."

Suppose Q_1Q_2 cuts P at O. Then we cannot connect Q_1 and Q_2 by any polygonal path without crossing the polygon.



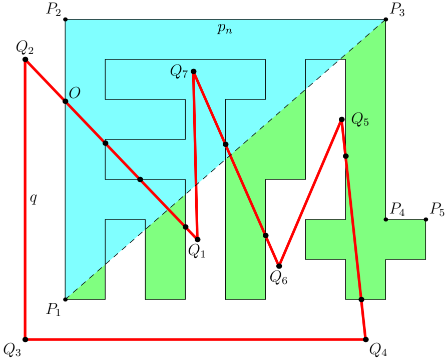
Veblen's Proof

Suppose polygon q intersects polygon p_n on P_1P_2 exactly once at O. We must find another point of intersection with another segment of P_n.



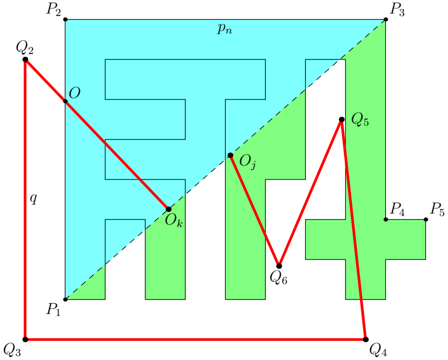
Veblen's Proof

q meets $P_1P_2P_3$ somewhere other than O. Suppose it meets on P_1P_3.

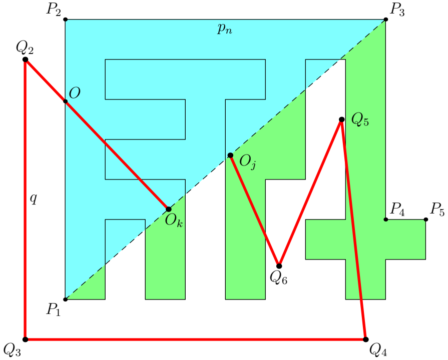


Veblen's Proof

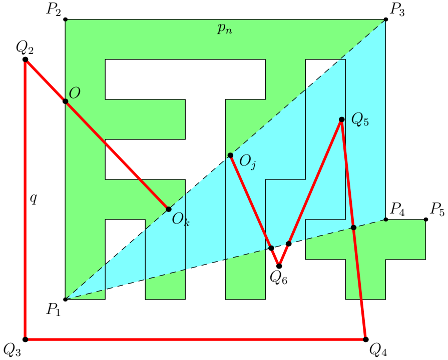
Obtain $O_k Q_2 Q_3 Q_4 Q_5 Q_6 O_j$, which "has a point inside and also a point outside the triangle $P_1 P_2 P_3$ and cuts the [triangle] $P_1 P_2 P_3$ only once." (my emphasis)



"Hence it has a point inside and a point outside any triangle of which P_1P_3 is a side."

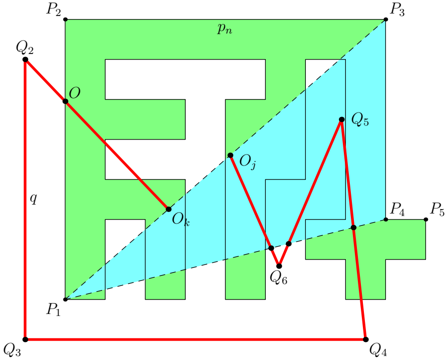


From this we conclude that $O_k Q_2 Q_3 Q_4 Q_5 Q_6 O_j$ cuts either $P_3 P_4$ or $P_1 P_4$.



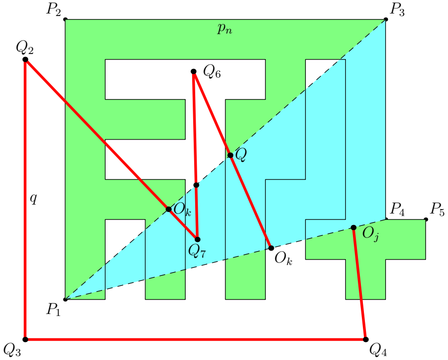
Veblen's Proof

"continuing this process"



Veblen's Proof

"continuing this process" ?



As it turns out...

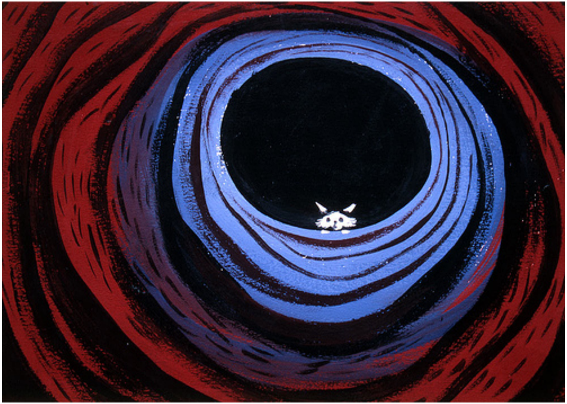
- According to Guggenheimer (citing Lennes and Hahn), the proof assumes the polygon can be triangulated and is only valid for convex polygons.

As it turns out...

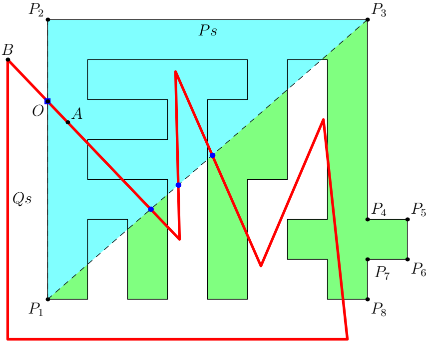
- According to Guggenheimer (citing Lennes and Hahn), the proof assumes the polygon can be triangulated and is only valid for convex polygons.
- According to Hahn, the proof is just "inconclusive".

As it turns out...

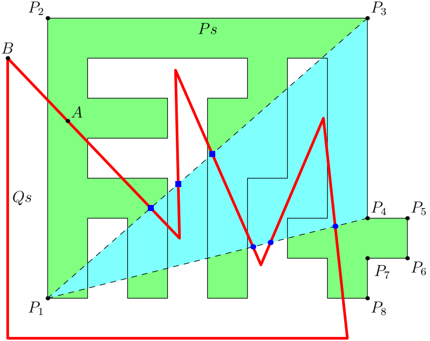
- According to Guggenheimer (citing Lennes and Hahn), the proof assumes the polygon can be triangulated and is only valid for convex polygons.
- According to Hahn, the proof is just "inconclusive".



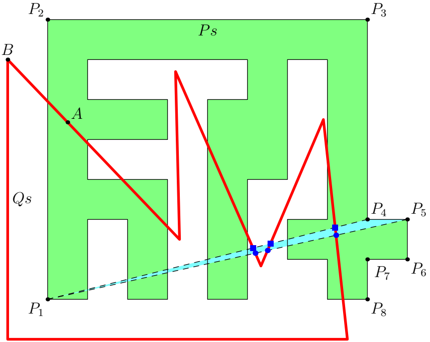
New Proof by Parity



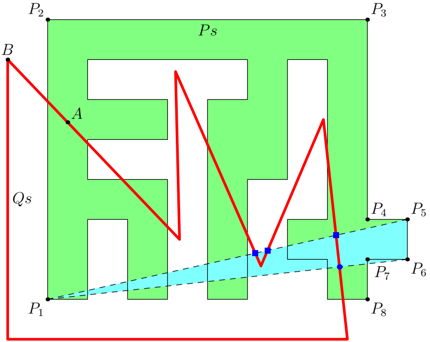
New Proof by Parity



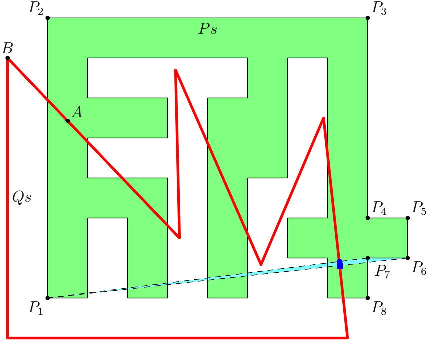
New Proof by Parity



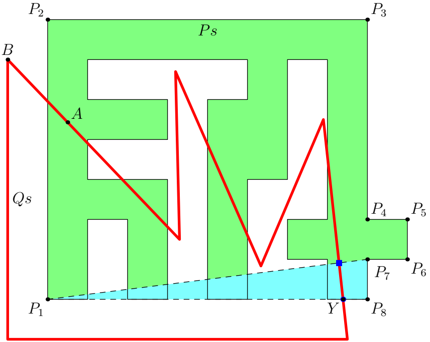
New Proof by Parity



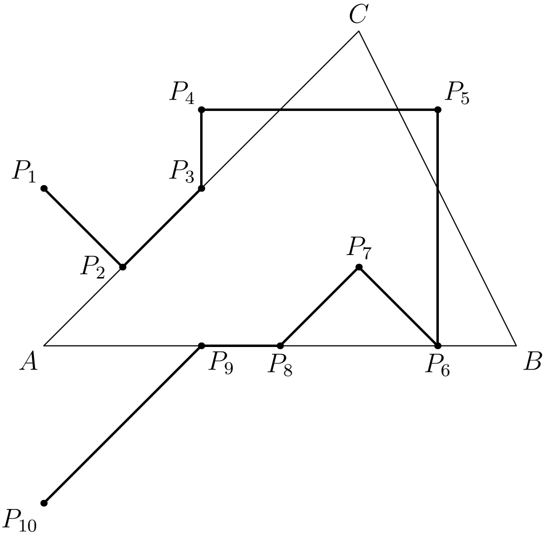
New Proof by Parity



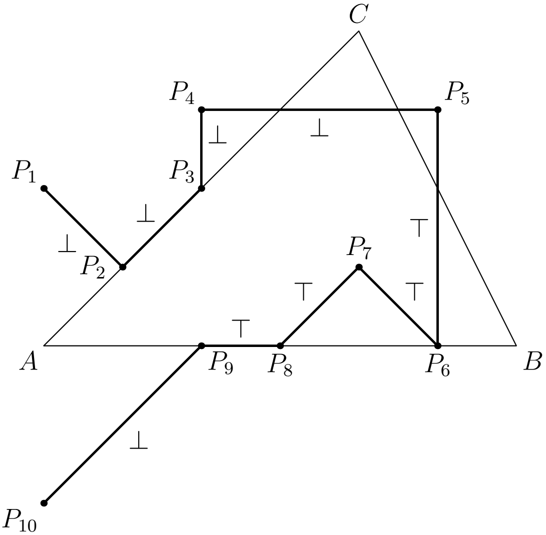
New Proof by Parity



Crossing a Triangle



Context (Γ : `bool`)



Definition of Crossings

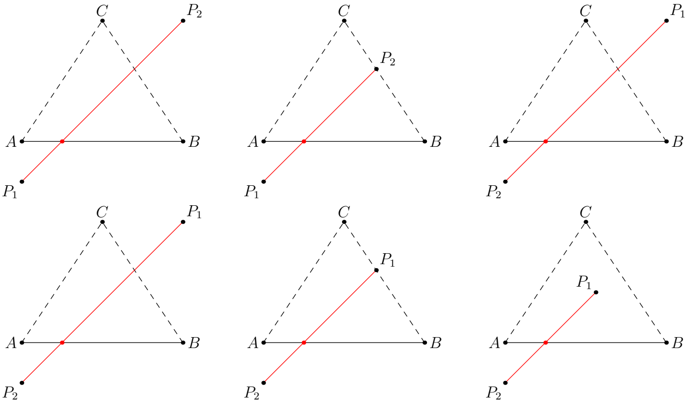
$$
\begin{aligned}
&\vdash_{def} \ \texttt{crossing}\ (A, B, C)\ \Gamma\ P_i\ P_{i+1} \\
&= \begin{cases}
0, & \text{if } \texttt{between}\ A\ P_i\ B \wedge \texttt{between}\ A\ P_{i+1}\ B \\
1, & \text{else if } \exists R.\ \texttt{between}\ P_i\ R\ P_{i+1} \wedge \texttt{between}\ A\ R\ B \\
1, & \text{else if } \texttt{between}\ A\ P_i\ B \\
 & \quad \wedge (\exists R.\ \texttt{between}\ P_i\ R\ P_{i+1} \\
 & \qquad\qquad \wedge\ \texttt{in\_triangle}\ (A, B, C)\ R \iff \neg\Gamma) \\
0, & \text{otherwise.}
\end{cases}
\end{aligned}
\tag{1}
$$

Definition of Context change

$$
\begin{aligned}
&\vdash_{def} \Gamma_{next}\ (A, B, C)\ \Gamma\ P_i\ P_{i+1} \\
&\iff \texttt{in\_triangle}\ (A, B, C)\ P_{i+1} \\
&\qquad \vee \left(\begin{array}{l} \texttt{on\_triangle}\ (A, B, C)\ P_{i+1} \\ \qquad \wedge \left(\begin{array}{l} (\exists R.\ \texttt{between}\ P_i\ R\ P_{i+1} \wedge \texttt{in\_triangle}\ (A, B, C)\ R) \\ \qquad \vee\ \texttt{on\_triangle}\ (A, B, C)\ P_i \wedge \Gamma) \end{array} \right) \end{array} \right).
\end{aligned}
$$

$$
\begin{aligned}
&\Gamma_{final}\ (A, B, C)\ \Gamma\ [] = \Gamma \\
&\Gamma_{final}\ (A, B, C)\ \Gamma\ ((P_i, P_{i+1}) : segments) = \\
&\qquad\qquad \Gamma_{final}\ (A, B, C)\ (\Gamma_{next}\ (A, B, C)\ \Gamma\ \ P_i\ P_{i+1})\ segments
\end{aligned}
$$

Crossings cases



A Crossings Lemma

$$\begin{aligned}
&\texttt{theorem}\ \neg(\exists a.\ \texttt{on_line}\ A\ a \wedge \texttt{on_line}\ B\ a \wedge \texttt{on_line}\ C\ a)\\
&\qquad \wedge \texttt{crossing}\ (A, B, C)\ X\ P_i\ P_{i+1} = 1\\
&\qquad \wedge \texttt{crossing}\ (A, C, B)\ X\ P_i\ P_{i+1} = 1\\
&\qquad \longrightarrow \texttt{crossing}\ (B, C, A)\ X\ P_i\ P_{i+1} = 0
\end{aligned}$$

$$\begin{aligned}
&\vdash \neg(\exists a.\ \texttt{on_line}\ A\ a \wedge \texttt{on_line}\ B\ a \wedge \texttt{on_line}\ C\ a)\\
&\quad \wedge \neg\texttt{on_polypath}\ [P_i, P_{i+1}]\ A \wedge \neg\texttt{on_polypath}\ [P_i, P_{i+1}]\ B\\
&\quad \wedge \neg\texttt{on_polypath}\ [P_i, P_{i+1}]\ C\\
&\quad \wedge (\neg\texttt{on_triangle}\ (A, B, C)\ P_i \longrightarrow (\texttt{in_triangle}\ (A, B, C)\ P_i \iff \Gamma))\\
&\longrightarrow \left(\begin{array}{l}
\texttt{crossing}\ (A, B, C)\ \Gamma\ P_i\ P_{i+1} + \texttt{crossing}\ (A, C, B)\ \Gamma\ P_i\ P_{i+1}\\
\qquad + \texttt{crossing}\ (B, C, A)\ \Gamma\ P_i\ P_{i+1} = 1\\
\iff \Gamma = \neg\Gamma_{next}\ (A, B, C)\ \Gamma\ P_i\ P_{i+1}
\end{array}\right)
\end{aligned}$$

Parity of Crossings

$$\vdash \texttt{polypath\_crossings}\ (A, B, C)\ \Gamma\ (\texttt{adjacent}\ Ps) > 0 \\ \longrightarrow \exists Q.\ \texttt{on\_polypath}\ Ps\ Q \wedge\ \texttt{between}\ A\ Q\ B$$

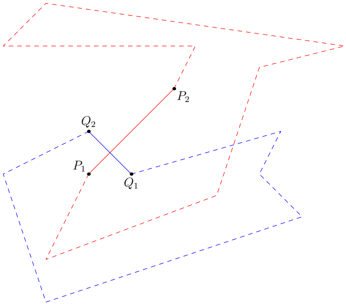
$$\begin{aligned}&\vdash Qs = [P] + Ps + [P]\\ &\quad \wedge \Gamma_{initial} = \Gamma_{final}\ (A, B, C)\ \Gamma\ (\texttt{adjacent}\ Qs)\\ &\quad \wedge \neg\texttt{on\_polypath}\ Qs\ A \wedge \neg\texttt{on\_polypath}\ Qs\ B \wedge \neg\texttt{on\_polypath}\ Qs\ C\\ &\quad \wedge \neg(\exists a.\ \texttt{on\_line}\ A\ a \wedge \texttt{on\_line}\ B\ a \wedge \texttt{on\_line}\ C\ a)\\ &\longrightarrow \texttt{even}\begin{pmatrix}\texttt{polypath\_crossings}\ (A, B, C)\ \Gamma_{initial}\ (\texttt{adjacent}\ Qs)\\ +\ \texttt{polypath\_crossings}\ (A, C, B)\ \Gamma_{initial}\ (\texttt{adjacent}\ Qs)\\ +\ \texttt{polypath\_crossings}\ (B, C, A)\ \Gamma_{initial}\ (\texttt{adjacent}\ Qs)\end{pmatrix}\end{aligned}$$

Moving a vertex (well-definedness)

$$
\begin{aligned}
&\vdash Qs = [P] + Ps + [P] \\
&\quad \wedge \neg \texttt{on\_polypath}\ Qs\ A \wedge \neg \texttt{on\_polypath}\ Qs\ B \\
&\quad \wedge \neg(\exists a.\ \texttt{on\_line}\ A\ a \wedge \texttt{on\_line}\ B\ a \wedge \texttt{on\_line}\ C\ a) \\
&\quad \wedge \neg(\exists a.\ \texttt{on\_line}\ A\ a \wedge \texttt{on\_line}\ B\ a \wedge \texttt{on\_line}\ C'\ a) \\
&\quad \longrightarrow \exists \Gamma'.\ \texttt{polypath\_crossings}\ (A, B, C) \\
&\qquad\qquad\qquad (\Gamma_{final}\ (A, B, C)\ \Gamma\ (\texttt{adjacent}\ Qs)) \\
&\qquad\qquad\qquad (\texttt{adjacent}\ Qs) \\
&\qquad\quad = \texttt{polypath\_crossings}\ (A, B, C') \\
&\qquad\qquad\qquad (\Gamma_{final}\ (A, B, C')\ \Gamma'\ (\texttt{adjacent}\ Qs)) \\
&\qquad\qquad\qquad (\texttt{adjacent}\ Qs)
\end{aligned}
$$

Verified Theorem

If two closed polygonal segments intersect at a point, then they meet again at another point.



Verified Theorem

$$\neg(\exists a.\texttt{on\_line}\ P_1\ a \wedge \texttt{on\_line}\ P_2\ a \wedge \texttt{on\_line}\ Q_1\ a \wedge \texttt{on\_line}\ Q_2\ a)$$

$$\texttt{between}\ P1\ X\ P2 \wedge \texttt{between}\ Q1\ X\ Q2$$

$$\longrightarrow \exists Y.\texttt{on\_polypath}\ (P2 : Ps) Y$$

$$\wedge\ \texttt{on\_polypath}\ (Q1 : Q2 : Qs)\ Y$$

$$\vee\ \texttt{on\_polypath}\ (Q2 : Qs)\ Y$$

$$\wedge\ \texttt{on\_polypath}\ (P1 : P2 : Ps)\ Y$$

Polygonal Jordan Curve Theorem: Part 2

There are at most two regions in the plane of a polygon which cannot be connected by a polygonal segment.

Polygonal Jordan Curve Theorem: Part 2

There are at most two regions in the plane of a polygon which cannot be connected by a polygonal segment.

- Consider a path which follows the edges of the polygon, staying close enough so as to avoid intersecting it.

Polygonal Jordan Curve Theorem: Part 2

There are at most two regions in the plane of a polygon which cannot be connected by a polygonal segment.

- Consider a path which follows the edges of the polygon, staying close enough so as to avoid intersecting it.
- But in our general setting, we cannot run paths parallel to the sides of the polygon.

Polygonal Jordan Curve Theorem: Part 2

There are at most two regions in the plane of a polygon which cannot be connected by a polygonal segment.

- Consider a path which follows the edges of the polygon, staying close enough so as to avoid intersecting it.
- But in our general setting, we cannot run paths parallel to the sides of the polygon.
- We cannot measure or compare distances (no ruler or compass)

Polygonal Jordan Curve Theorem: Part 2

There are at most two regions in the plane of a polygon which cannot be connected by a polygonal segment.

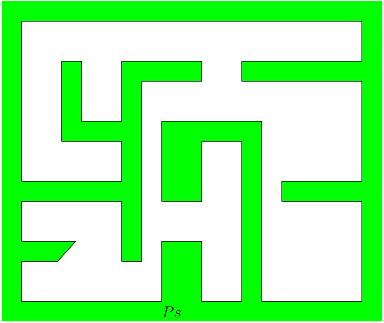
- Consider a path which follows the edges of the polygon, staying close enough so as to avoid intersecting it.
- But in our general setting, we cannot run paths parallel to the sides of the polygon.
- We cannot measure or compare distances (no ruler or compass)
- We cannot reason about our orientation via angles.

Polygonal Jordan Curve Theorem: Part 2

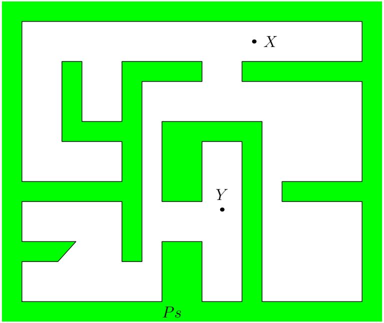
There are at most two regions in the plane of a polygon which cannot be connected by a polygonal segment.

- Consider a path which follows the edges of the polygon, staying close enough so as to avoid intersecting it.
- But in our general setting, we cannot run paths parallel to the sides of the polygon.
- We cannot measure or compare distances (no ruler or compass)
- We cannot reason about our orientation via angles.
- How do we squeeze through corridors in the maze which might be infinitesimally narrow?

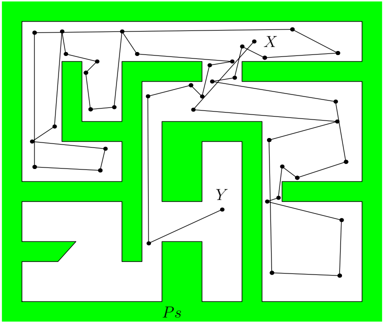
Polygonal JCT Part 2: Proof



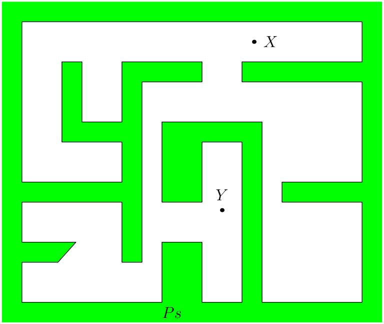
Polygonal JCT Part 2: Proof



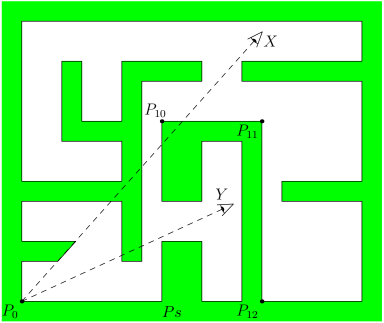
Polygonal JCT Part 2: Proof



Lines-of-sight

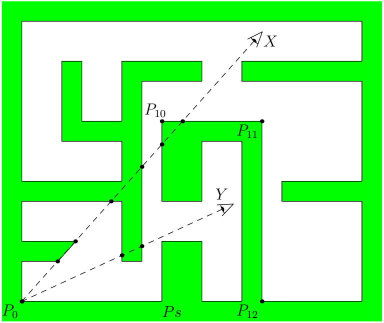


Where are we?



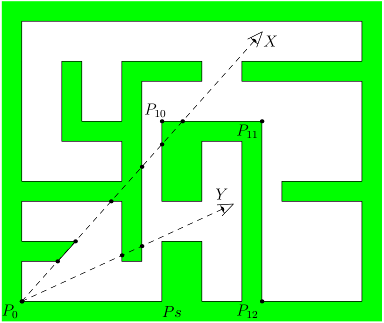
Let's see: consider all the points between P_0 and X and P_0 and Y

Lines-of-sight

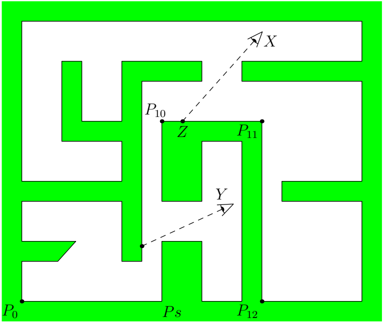


Now pick out the intersections along the polygon's path.

Lines-of-sight



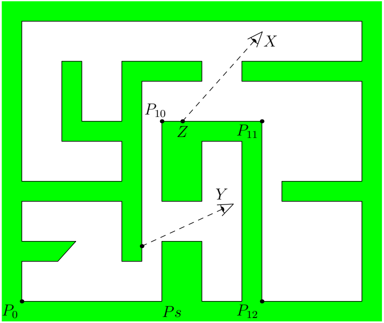
Use `ORDER_TAC`



Raycast

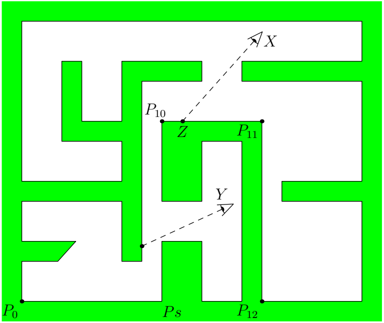
$$
\begin{aligned}
&\forall Ps\ X\ P_0.\ \neg\texttt{on\_polypath}\ Ps\ X \wedge \texttt{on\_polypath}\ Ps\ P_0 \\
&\qquad \longrightarrow \exists Z.\ \texttt{on\_polypath}\ Ps\ Z \wedge (\texttt{between}\ X\ Z\ P_0 \vee P_0 = Z) \\
&\qquad\qquad\qquad\qquad \wedge \neg(\exists R.\ \texttt{between}\ X\ R\ Z \wedge \texttt{on\_polypath}\ Ps\ R)
\end{aligned}
$$

Lines-of-sight



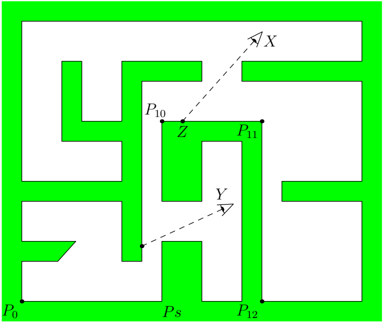
X has *line-of-sight* to the point Z on edge $P_{10}P_{11}$.

First Move: Navigating a Local Concavity



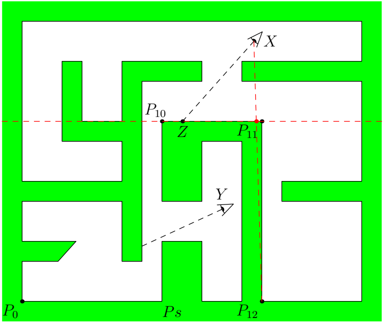
We now want to navigate so that X has line-of-sight to the next edge: $P_{11}P_{12}$.

First Move: Navigating a Local Concavity



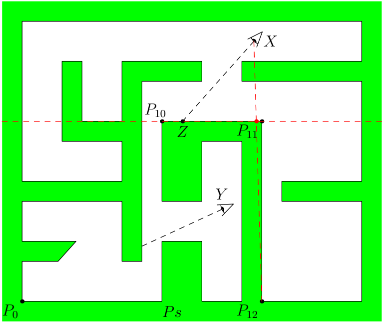
X and P_{12} are on *opposite sides* of the line $P_{10}P_{11}$.

First Move: Navigating a Local Concavity



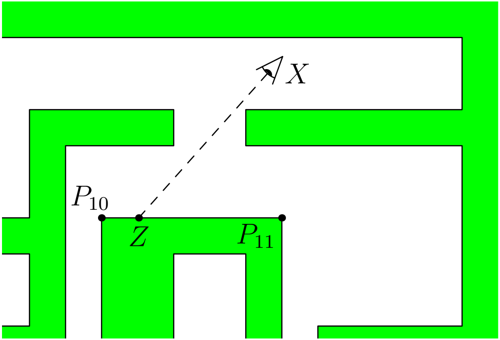
X and P_{12} are on *opposite sides* of the line $P_{10}P_{11}$.
Formally, there is a point between X and P_{12} on the line $P_{10}P_{11}$.

First Move: Navigating a Local Concavity



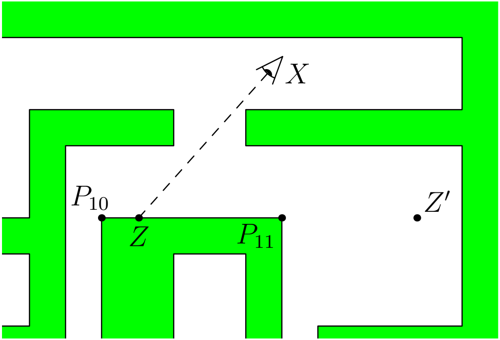
X and P_{12} are on *opposite sides* of the line $P_{10}P_{11}$.
Formally, there is a point between X and P_{12} on the line $P_{10}P_{11}$.

First Move: Navigating a Local Concavity



Let's take a closer look.

First Move: Navigating a Local Concavity

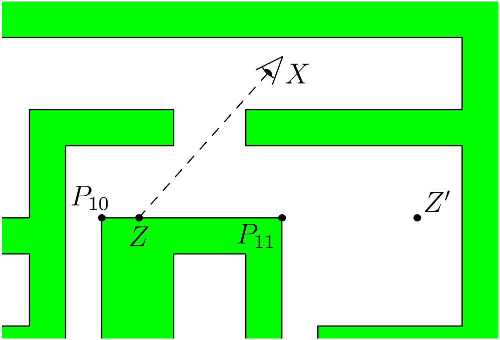


Find some Z' in the direction $P_{10}P_{11}$ by Axiom II,2

$$\vdash P_{10} \neq P_{11} \longrightarrow \exists Z'.\ \texttt{between}\ P_{10}\ P_{11}\ Z'$$

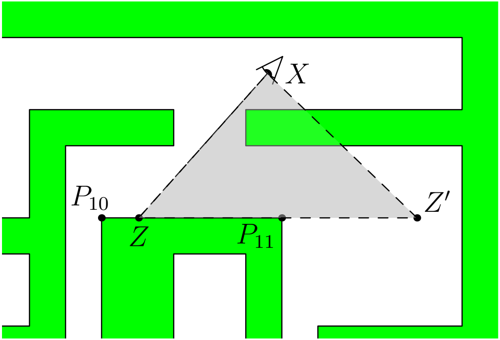
(raycast if necessary).

First Move: Navigating a Local Concavity



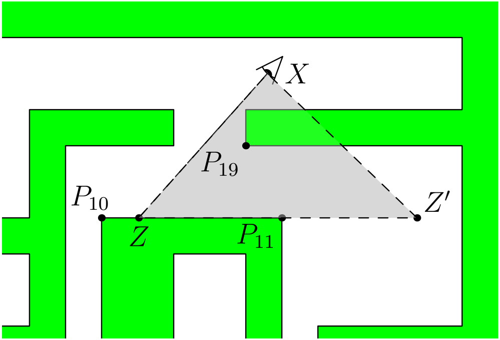
X does not have line-of-sight to Z'

First Move: Navigating a Local Concavity



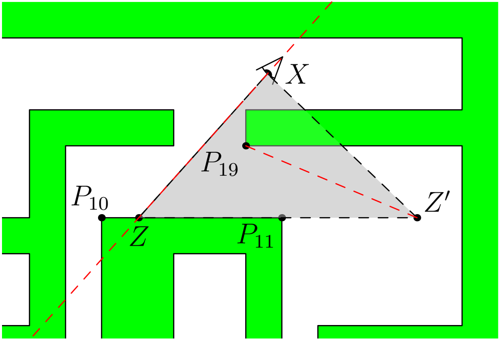
X does not have line-of-sight to Z'
Let's draw the triangle $ZZ'X$.

First Move: Navigating a Local Concavity



Note that the first point which is *inside* this triangle is P_{19}.

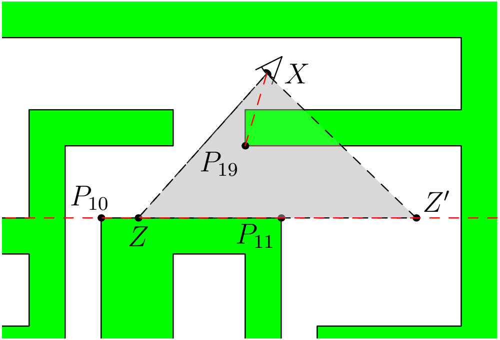
First Move: Navigating a Local Concavity



Formally,

1. there is no point between P_{19} and Z' on the line XZ;

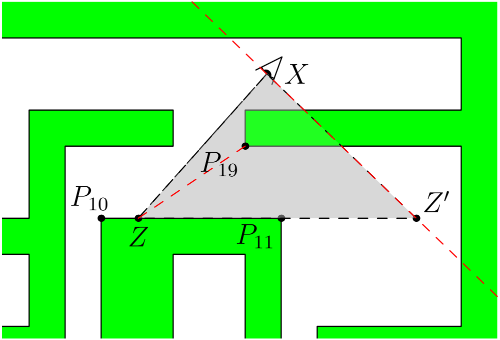
First Move: Navigating a Local Concavity



Formally,

1. there is no point between P_{19} and Z' on the line XZ;
2. there is no point between P_{19} and X on the line ZZ';

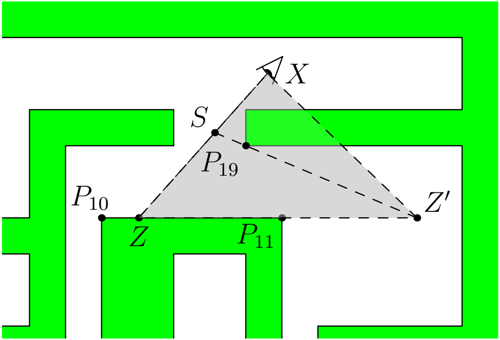
First Move: Navigating a Local Concavity



Formally,

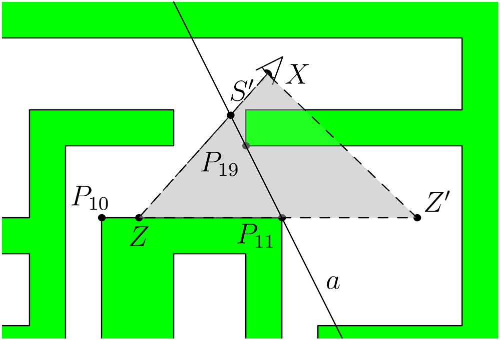
1. there is no point between P_{19} and Z' on the line XZ;
2. there is no point between P_{19} and X on the line ZZ';
3. there is no point between P_{19} and Z on the line XZ';

First Move: Navigating a Local Concavity



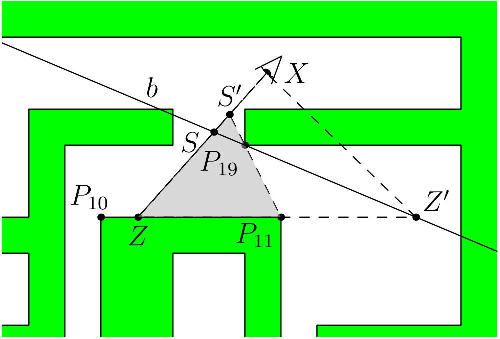
We now want to find the point S where $Z'P_{19}$ meets XZ.

First Move: Navigating a Local Concavity



We now want to find the point S where $Z'P_{19}$ meets XZ.
Use Pasch's Axiom once.

First Move: Navigating a Local Concavity

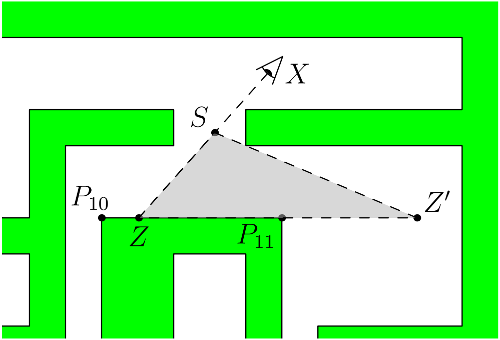


We now want to find the point S where $Z'P_{19}$ meets XZ.

Use Pasch's Axiom once.

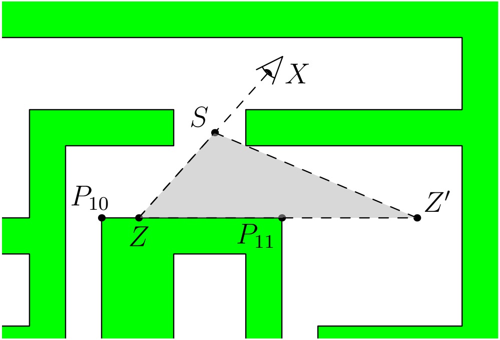
And once more.

First Move: Navigating a Local Concavity



We now look at the triangle $ZZ'S$

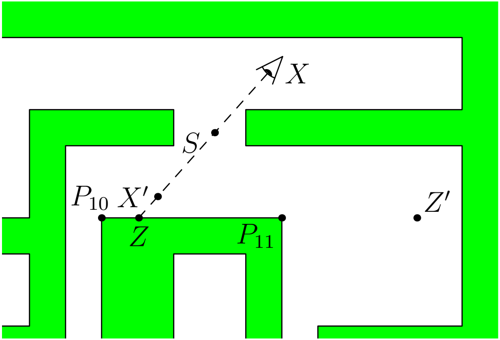
First Move: Navigating a Local Concavity



We now look at the triangle $ZZ'S$

It doesn't contain any points of the polygon. So any lines between its edges are lines-of-sight.

First Move: Navigating a Local Concavity

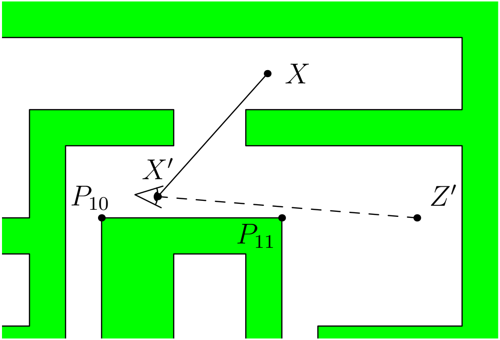


We now look at the triangle $ZZ'S$

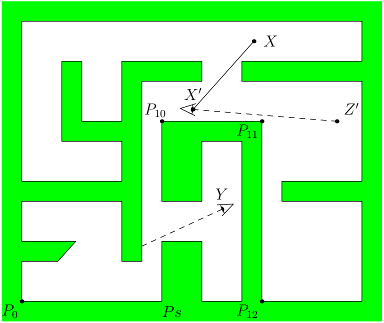
It doesn't contain any points of the polygon. So any lines between its edges are lines-of-sight.

So pick a point X' between S' and Z (Hilbert's Theorem 4).

First Move: Navigating a Local Concavity

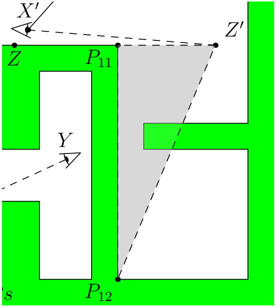


First Move: Navigating a Local Concavity

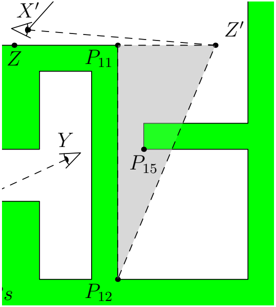


We still need line-of-sight to a point on $P_{11}P_{12}$.

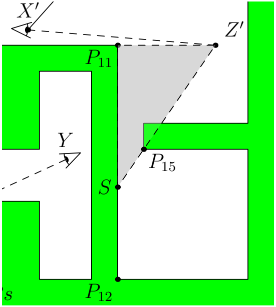
First Move: Navigating a Local Concavity



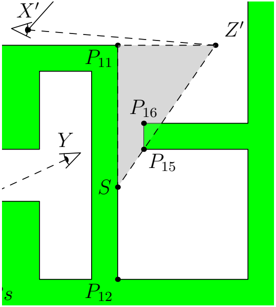
First Move: Navigating a Local Concavity



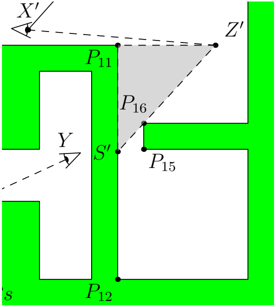
First Move: Navigating a Local Concavity



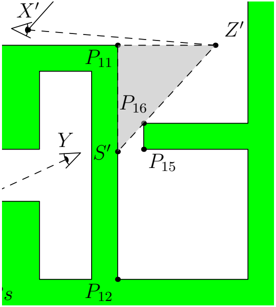
First Move: Navigating a Local Concavity



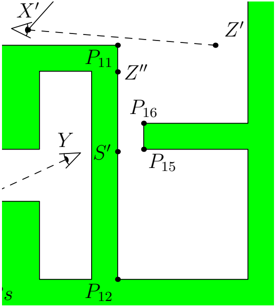
First Move: Navigating a Local Concavity



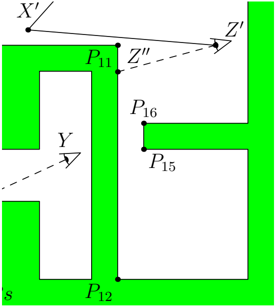
First Move: Navigating a Local Concavity



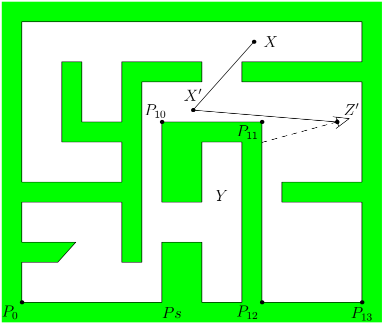
First Move: Navigating a Local Concavity



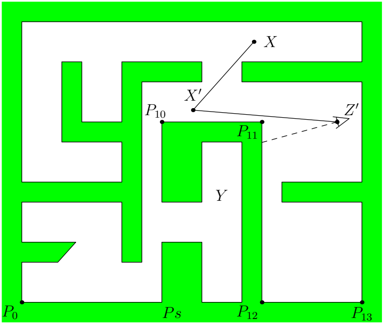
First Move: Navigating a Local Concavity



Second Move: Navigating a Local Convexity



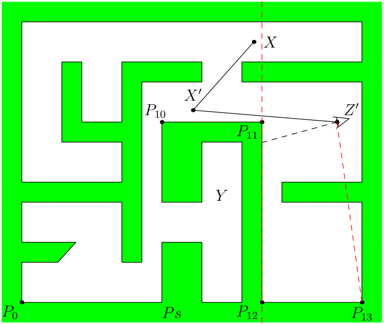
We now want to navigate so that Z' has line-of-sight to the next edge: $P_{12}P_{13}$.



We now want to navigate so that Z' has line-of-sight to the next edge: $P_{12}P_{13}$.

Formally, Z' and P_{13} are on the same side of the line $P_{11}P_{12}$.

Second Move: Navigating a Local Convexity

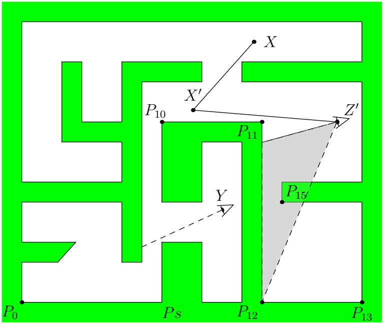


We now want to navigate so that Z' has line-of-sight to the next edge: $P_{12}P_{13}$.

Formally, Z' and P_{13} are on the same side of the line $P_{11}P_{12}$.

More formally, there is no point between Z' and P_{13} on the line $P_{11}P_{12}$.

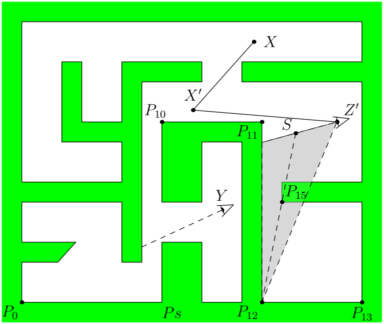
Second Move: Navigating a Local Convexity



Squeeze!

$$
\begin{aligned}
&\vdash \neg \texttt{on\_polypath}\ (Ps - [P_{11}, P_{12}])\ P_{11} \\
&\quad \wedge \neg(\exists X.\ \texttt{between}\ Z'\ X\ P_{11} \wedge \texttt{on\_polypath}\ (Ps - [P_{11}, P_{12}])\ X) \\
&\quad \wedge \neg(\exists X.\ \texttt{between}\ P_{11}\ X\ P_{12} \wedge \texttt{on\_polypath}\ (Ps - [P_{11}, P_{12}])\ X) \\
&\quad \longrightarrow \exists S'.\ \texttt{between}\ Z'\ S'\ P_{11} \\
&\qquad \wedge \neg \exists X.\ \texttt{in\_triangle}\ (S', P_{11}, P_{12})\ X \wedge \texttt{on\_polypath}\ (Ps - [P_{11}, P_{12}])\ X.
\end{aligned}
$$

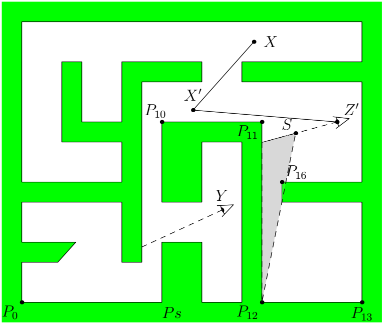
Second Move: Navigating a Local Convexity



Squeeze!

$$\vdash \neg \texttt{on\_polypath}\ (Ps - [P_{11}, P_{12}])\ P_{11}$$
$$\wedge \neg(\exists X.\ \texttt{between}\ Z'\ X\ P_{11} \wedge \texttt{on\_polypath}\ (Ps - [P_{11}, P_{12}])\ X)$$
$$\wedge \neg(\exists X.\ \texttt{between}\ P_{11}\ X\ P_{12} \wedge \texttt{on\_polypath}\ (Ps - [P_{11}, P_{12}])\ X)$$
$$\longrightarrow \exists S'.\ \texttt{between}\ Z'\ S'\ P_{11}$$
$$\wedge \neg \exists X.\ \texttt{in\_triangle}\ (S', P_{11}, P_{12})\ X \wedge \texttt{on\_polypath}\ (Ps - [P_{11}, P_{12}])\ X.$$

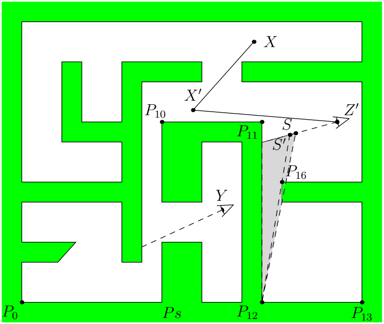
Second Move: Navigating a Local Convexity



Squeeze!

$$\begin{aligned}
&\vdash \neg \mathtt{on_polypath}\ (Ps - [P_{11}, P_{12}])\ P_{11} \\
&\quad \wedge \neg(\exists X.\ \mathtt{between}\ Z'\ X\ P_{11} \wedge \mathtt{on_polypath}\ (Ps - [P_{11}, P_{12}])\ X) \\
&\quad \wedge \neg(\exists X.\ \mathtt{between}\ P_{11}\ X\ P_{12} \wedge \mathtt{on_polypath}\ (Ps - [P_{11}, P_{12}])\ X) \\
&\quad \longrightarrow \exists S'.\ \mathtt{between}\ Z'\ S'\ P_{11} \\
&\qquad \wedge \neg\exists X.\ \mathtt{in_triangle}\ (S', P_{11}, P_{12})\ X \wedge \mathtt{on_polypath}\ (Ps - [P_{11}, P_{12}])\ X.
\end{aligned}$$

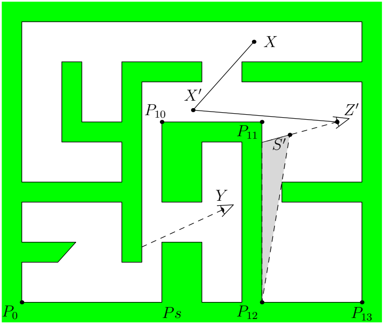
Second Move: Navigating a Local Convexity



Squeeze!

$$\vdash \neg \texttt{on\_polypath}\ (Ps - [P_{11}, P_{12}])\ P_{11}$$
$$\wedge \neg(\exists X.\ \texttt{between}\ Z'\ X\ P_{11} \wedge \texttt{on\_polypath}\ (Ps - [P_{11}, P_{12}])\ X)$$
$$\wedge \neg(\exists X.\ \texttt{between}\ P_{11}\ X\ P_{12} \wedge \texttt{on\_polypath}\ (Ps - [P_{11}, P_{12}])\ X)$$
$$\longrightarrow \exists S'.\ \texttt{between}\ Z'\ S'\ P_{11}$$
$$\wedge \neg\exists X.\ \texttt{in\_triangle}\ (S', P_{11}, P_{12})\ X \wedge \texttt{on\_polypath}\ (Ps - [P_{11}, P_{12}])\ X.$$

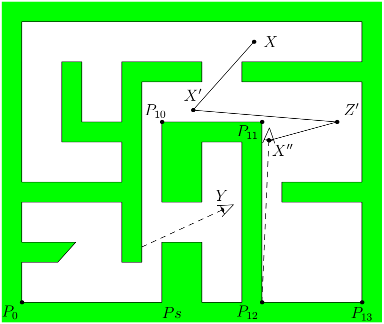
Second Move: Navigating a Local Convexity



Squeeze!

$$\vdash \neg \texttt{on\_polypath}\ (Ps - [P_{11}, P_{12}])\ P_{11}$$
$$\wedge \neg(\exists X.\ \texttt{between}\ Z'\ X\ P_{11} \wedge \texttt{on\_polypath}\ (Ps - [P_{11}, P_{12}])\ X)$$
$$\wedge \neg(\exists X.\ \texttt{between}\ P_{11}\ X\ P_{12} \wedge \texttt{on\_polypath}\ (Ps - [P_{11}, P_{12}])\ X)$$
$$\longrightarrow \exists S'.\ \texttt{between}\ Z'\ S'\ P_{11}$$
$$\wedge \neg\exists X.\ \texttt{in\_triangle}\ (S', P_{11}, P_{12})\ X \wedge \texttt{on\_polypath}\ (Ps - [P_{11}, P_{12}])\ X.$$

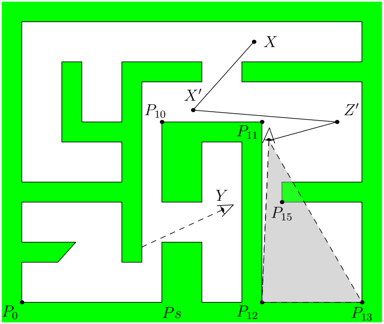
Second Move: Navigating a Local Convexity



Squeeze!

$$\begin{aligned}
&\vdash \neg \texttt{on_polypath}\ (Ps - [P_{11}, P_{12}])\ P_{11} \\
&\quad \wedge \neg(\exists X.\ \texttt{between}\ Z'\ X\ P_{11} \wedge \texttt{on_polypath}\ (Ps - [P_{11}, P_{12}])\ X) \\
&\quad \wedge \neg(\exists X.\ \texttt{between}\ P_{11}\ X\ P_{12} \wedge \texttt{on_polypath}\ (Ps - [P_{11}, P_{12}])\ X) \\
&\quad \longrightarrow \exists S'.\ \texttt{between}\ Z'\ S'\ P_{11} \\
&\qquad\quad \wedge \neg\exists X.\ \texttt{in_triangle}\ (S', P_{11}, P_{12})\ X \wedge \texttt{on_polypath}\ (Ps - [P_{11}, P_{12}])\ X.
\end{aligned}$$

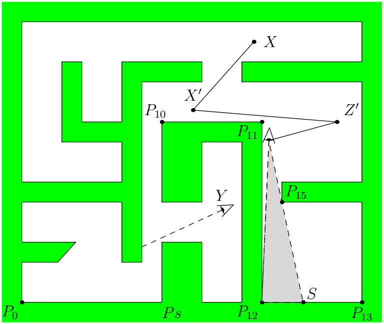
Second Move: Navigating a Local Convexity



Squeeze!

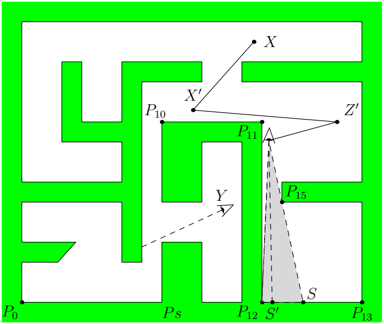
$$
\begin{aligned}
&\vdash \neg(\texttt{on\_polypath}\ (Ps - [P_{12}, P_{13}]) P_{12} \\
&\quad \wedge \neg(\exists X.\ \texttt{between}\ P_{13}\ X\ P_{12} \wedge \texttt{on\_polypath}\ (Ps - [P_{12}, P_{13}])\ X) \\
&\quad \wedge \neg(\exists X.\ \texttt{between}\ P_{12}\ X\ P_{11} \wedge \texttt{on\_polypath}\ (Ps - [P_{12}, P_{13}])\ X) \\
&\quad \longrightarrow \exists Z''.\ \texttt{between}\ P_{13}\ Z''\ P_{12} \\
&\qquad\quad \wedge \neg\exists X.\ \texttt{in\_triangle}\ (Z'', P_{12}, P_{11})\ X \wedge \texttt{on\_polypath}\ (P - [P_{12}, P_{13}])\ X.
\end{aligned}
$$

Second Move: Navigating a Local Convexity



Squeeze!

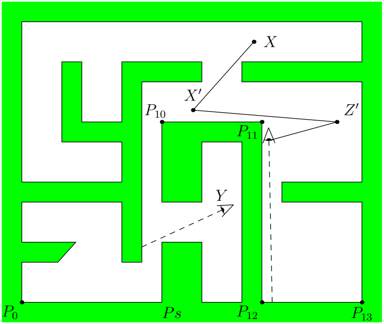
$$
\begin{aligned}
\vdash \neg(&\texttt{on\_polypath}\ (Ps - [P_{12}, P_{13}]) P_{12} \\
&\wedge \neg(\exists X.\ \texttt{between}\ P_{13}\ X\ P_{12} \wedge \texttt{on\_polypath}\ (Ps - [P_{12}, P_{13}])\ X) \\
&\wedge \neg(\exists X.\ \texttt{between}\ P_{12}\ X\ P_{11} \wedge \texttt{on\_polypath}\ (Ps - [P_{12}, P_{13}])\ X) \\
&\longrightarrow \exists Z''.\ \texttt{between}\ P_{13}\ Z''\ P_{12} \\
&\qquad \wedge \neg\exists X.\ \texttt{in\_triangle}\ (Z'', P_{12}, P_{11})\ X \wedge \texttt{on\_polypath}\ (P - [P_{12}, P_{13}])\ X.
\end{aligned}
$$



Squeeze!

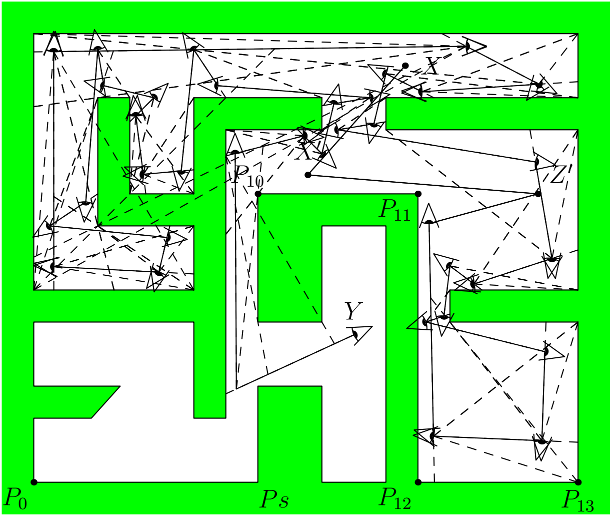
$$\begin{aligned}
&\vdash \neg(\texttt{on_polypath}\ (Ps - [P_{12}, P_{13}]) P_{12} \\
&\quad \wedge \neg(\exists X.\ \texttt{between}\ P_{13}\ X\ P_{12} \wedge \texttt{on_polypath}\ (Ps - [P_{12}, P_{13}])\ X) \\
&\quad \wedge \neg(\exists X.\ \texttt{between}\ P_{12}\ X\ P_{11} \wedge \texttt{on_polypath}\ (Ps - [P_{12}, P_{13}])\ X) \\
&\quad \longrightarrow \exists Z''.\ \texttt{between}\ P_{13}\ Z''\ P_{12} \\
&\qquad\quad \wedge \neg\exists X.\ \texttt{in_triangle}\ (Z'', P_{12}, P_{11})\ X \wedge \texttt{on_polypath}\ (P - [P_{12}, P_{13}])\ X.
\end{aligned}$$

Second Move: Navigating a Local Convexity

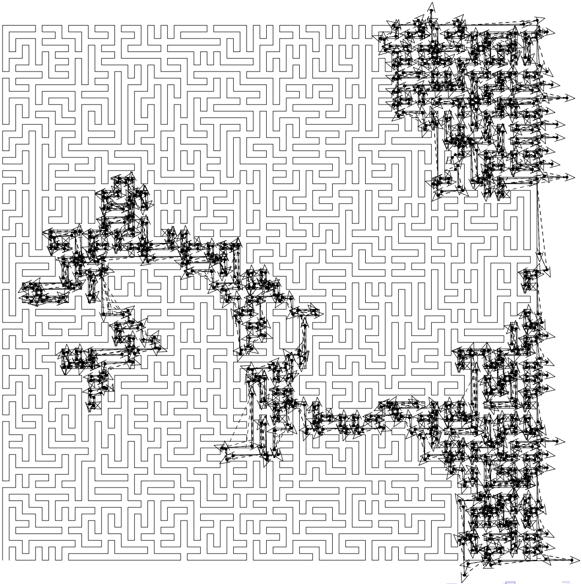


Squeeze!

$$\vdash \neg(\texttt{on\_polypath}\ (Ps - [P_{12}, P_{13}]) P_{12}$$
$$\wedge \neg(\exists X.\ \texttt{between}\ P_{13}\ X\ P_{12} \wedge \texttt{on\_polypath}\ (Ps - [P_{12}, P_{13}])\ X)$$
$$\wedge \neg(\exists X.\ \texttt{between}\ P_{12}\ X\ P_{11} \wedge \texttt{on\_polypath}\ (Ps - [P_{12}, P_{13}])\ X)$$
$$\longrightarrow \exists Z''.\ \texttt{between}\ P_{13}\ Z''\ P_{12}$$
$$\wedge \neg\exists X.\ \texttt{in\_triangle}\ (Z'', P_{12}, P_{11})\ X \wedge \texttt{on\_polypath}\ (P - [P_{12}, P_{13}])\ X.$$


X
X
Z
P_{10}
P_{11}
Y
P_0
Ps
P_{12}
P_{13}

And again



Edge-to-edge

$$
\begin{aligned}
&\vdash \texttt{between}\ P_1\ X'\ P_2 \wedge P_2 \neq P_3 \\
&\quad \wedge \neg \texttt{on\_polypath}\ (P_1 : P_2 : P_3 : Ps)\ X \wedge \neg \texttt{on\_polypath}\ (P_3 : Ps)\ P_2 \\
&\quad \wedge \neg (\exists Z.\ \texttt{between}\ X\ Z\ X' \wedge \texttt{on\_polypath}\ (P_1 : P_2 : P_3 : Ps)\ Z) \\
&\quad \wedge \neg (\exists Z.\ \texttt{between}\ P_1\ Z\ P_2 \wedge \texttt{on\_polypath}\ (P_2 : P_3 : Ps)\ Z) \\
&\quad \wedge \neg (\exists Z.\ \texttt{between}\ P_2\ Z\ P_3 \wedge \texttt{on\_polypath}\ (P_3 : Ps)\ Z) \\
&\quad \longrightarrow \exists Y.\ \exists Y'. \\
&\qquad\quad \texttt{polypath\_connected}\ (\texttt{on\_polypath}\ (P_1 : P_2 : P_3 : Ps))\ X\ Y \\
&\qquad\quad \wedge \texttt{between}\ P_2\ Y'\ P_3 \wedge \neg \texttt{on\_polypath}\ (P_1 : P_2 : P_3 : Ps)\ Y) \\
&\qquad\quad \wedge \neg \exists Z.\ \texttt{between}\ Y\ Z\ Y' \wedge (P_1 : P_2 : P_3 : Ps)\ Z).
\end{aligned}
$$

Moving to the first edge

$$
\begin{aligned}
\vdash & \texttt{simple\_polygon}\ Ps \wedge \neg\texttt{on\_polypath}\ Ps\ X \\
& \wedge \texttt{mem}\ (P, Q)\ (\texttt{adjacent}\ Ps) \wedge \texttt{between}\ P\ X'\ Q \\
& \wedge \neg(\exists Z.\ \texttt{between}\ X\ Z\ X' \wedge \texttt{on\_polypath}\ Ps\ Z) \\
& \qquad \longrightarrow \exists Y.\ \exists Y'.\ \texttt{polypath\_connected}\ (\texttt{on\_polypath}\ Ps)\ X\ Y \\
& \qquad\qquad\qquad \wedge \neg\texttt{on\_polypath}\ Ps\ Y \\
& \qquad\qquad\qquad \wedge \texttt{between}\ (\texttt{head}\ Ps)\ Y'\ (\texttt{head}\ (\texttt{tail}\ Ps)) \\
& \qquad\qquad\qquad \wedge \neg\exists Z.\ \texttt{between}\ Y\ Z\ Y' \wedge \texttt{on\_polypath}\ Ps\ Z.
\end{aligned}
$$

Polygonal Jordan Curve Theorem

$$\begin{aligned}
\vdash & \texttt{simple_polygon}\ Ps \\
& \longrightarrow \exists P.\ \exists Q.\ \neg\texttt{on_polypath}\ Ps\ P \wedge \neg\texttt{on_polypath}\ Ps\ Q \\
& \qquad \wedge \neg\texttt{polypath_connected}\ (\texttt{on_polypath}\ Ps)\ P\ Q
\end{aligned}$$

$$\begin{aligned}
\vdash & \texttt{simple_polygon}\ Ps \\
& \wedge \neg\texttt{on_polypath}\ Ps\ P \wedge \neg\texttt{on_polypath}\ Ps\ Q \wedge \neg\texttt{on_polypath}\ Ps\ R \\
& \longrightarrow \texttt{polypath_connected}\ (\texttt{on_polypath}\ Ps)\ P\ Q \\
& \qquad \vee \texttt{polypath_connected}\ (\texttt{on_polypath}\ Ps)\ P\ R \\
& \qquad \vee \texttt{polypath_connected}\ (\texttt{on_polypath}\ Ps)\ Q\ R
\end{aligned}$$

Polygonal Jordan Curve Theorem

$$
\begin{aligned}
&\vdash \texttt{simple\_polygon}\ Ps \\
&\quad \longrightarrow \exists P.\ \exists Q.\ \neg\texttt{on\_polypath}\ Ps\ P \wedge \neg\texttt{on\_polypath}\ Ps\ Q \\
&\qquad \wedge \neg\texttt{polypath\_connected}\ (\texttt{on\_polypath}\ Ps)\ P\ Q
\end{aligned}
$$

$$
\begin{aligned}
&\vdash \texttt{simple\_polygon}\ Ps \\
&\quad \wedge \neg\texttt{on\_polypath}\ Ps\ P \wedge \neg\texttt{on\_polypath}\ Ps\ Q \wedge \neg\texttt{on\_polypath}\ Ps\ R \\
&\quad \longrightarrow \texttt{polypath\_connected}\ (\texttt{on\_polypath}\ Ps)\ P\ Q \\
&\qquad \vee \texttt{polypath\_connected}\ (\texttt{on\_polypath}\ Ps)\ P\ R \\
&\qquad \vee \texttt{polypath\_connected}\ (\texttt{on\_polypath}\ Ps)\ Q\ R
\end{aligned}
$$

No subgoals

Polygonal Jordan Curve Theorem

$$\begin{aligned} &\vdash \texttt{simple\_polygon}\ Ps \\ &\quad \longrightarrow \exists P.\ \exists Q.\ \neg \texttt{on\_polypath}\ Ps\ P \wedge \neg \texttt{on\_polypath}\ Ps\ Q \\ &\qquad\quad \wedge \neg \texttt{polypath\_connected}\ (\texttt{on\_polypath}\ Ps)\ P\ Q \end{aligned}$$

$$\begin{aligned} &\vdash \texttt{simple\_polygon}\ Ps \\ &\quad \wedge \neg \texttt{on\_polypath}\ Ps\ P \wedge \neg \texttt{on\_polypath}\ Ps\ Q \wedge \neg \texttt{on\_polypath}\ Ps\ R \\ &\quad \longrightarrow \texttt{polypath\_connected}\ (\texttt{on\_polypath}\ Ps)\ P\ Q \\ &\qquad\quad \vee \texttt{polypath\_connected}\ (\texttt{on\_polypath}\ Ps)\ P\ R \\ &\qquad\quad \vee \texttt{polypath\_connected}\ (\texttt{on\_polypath}\ Ps)\ Q\ R \end{aligned}$$

`No subgoals(!)`

Final Notes

- Axioms here define what is sometimes called *Ordered Geometry*.

Final Notes

- Axioms here define what is sometimes called *Ordered Geometry*.
- "It is astonishing that none of the textbooks of elementary axiomatic geometry gives a proof [of the Polygonal Jordan Curve Theorem from Ordered Geometry]" — Guggenheimer

Final Notes

- Axioms here define what is sometimes called *Ordered Geometry*.
- "It is astonishing that none of the textbooks of elementary axiomatic geometry gives a proof [of the Polygonal Jordan Curve Theorem from Ordered Geometry]" — Guggenheimer
- Now formally verified in a "readable" style.

The Future of Verified Mathematics

$$
\begin{array}{l}
\forall e. e > 0 \\
\quad \longrightarrow \texttt{FINITE} \left\{ \begin{array}{l}
(a, b, c) \mid \texttt{coprime}\,(a, b) \\
\qquad \wedge\ \texttt{coprime}\,(a, c) \\
\qquad \wedge\ \texttt{coprime}\,(b, c) \\
\qquad \wedge\ a + b = c \\
\qquad \wedge\ c > \texttt{ITSET}\,(\times) \\
\qquad\quad \{p \mid \texttt{prime}\, p \wedge p\,\texttt{divides}\,(a \times b \times c)\})\, 1 \\
\qquad\qquad \texttt{EXP}\,(1 + e)
\end{array} \right\}
\end{array}
$$

The Future of Verified Mathematics

$$\forall e. e > 0$$

$$\longrightarrow \texttt{FINITE} \left\{ \begin{array}{l} (a, b, c) \mid \texttt{coprime}\,(a, b) \\ \quad \wedge\ \texttt{coprime}\,(a, c) \\ \quad \wedge\ \texttt{coprime}\,(b, c) \\ \quad \wedge\ a + b = c \\ \quad \wedge\ c > \texttt{ITSET}\,(\times) \\ \qquad \{p \mid \texttt{prime}\; p \wedge p\, \texttt{divides}\,(a \times b \times c)\})\, 1 \\ \qquad\quad \texttt{EXP}\,(1 + e) \end{array} \right\}$$

$$\vdots$$

`No subgoals (?)`