

Isabelle/HOL Exercises

Logic and Sets

Propositional Logic

In this exercise, we will prove some lemmas of propositional logic with the aid of a calculus of natural deduction.

For the proofs, you may only use

- the following lemmas:
 $notI: (A \implies False) \implies \neg A,$
 $notE: [\neg A; A] \implies B,$
 $conjI: [A; B] \implies A \wedge B,$
 $conjE: [A \wedge B; [A; B] \implies C] \implies C,$
 $disjI1: A \implies A \vee B,$
 $disjI2: A \implies B \vee A,$
 $disjE: [A \vee B; A \implies C; B \implies C] \implies C,$
 $impI: (A \implies B) \implies A \longrightarrow B,$
 $impE: [A \longrightarrow B; A; B \implies C] \implies C,$
 $mp: [A \longrightarrow B; A] \implies B$
 $iffI: [A \implies B; B \implies A] \implies A = B,$
 $iffE: [A = B; [A \longrightarrow B; B \longrightarrow A] \implies C] \implies C$
 $classical: (\neg A \implies A) \implies A$
- the proof methods *rule*, *erule* and *assumption*.

Prove:

```
lemma I: "A  $\longrightarrow$  A"  
  apply (rule impI)  
  apply assumption  
done
```

```
lemma "A  $\wedge$  B  $\longrightarrow$  B  $\wedge$  A"  
  apply (rule impI)  
  apply (erule conjE)  
  apply (rule conjI)  
  apply assumption  
  apply assumption
```

done

lemma " $(A \wedge B) \longrightarrow (A \vee B)$ "

 apply (rule impI)
 apply (erule conjE)
 apply (rule disjI1)
 apply assumption

done

lemma " $((A \vee B) \vee C) \longrightarrow A \vee (B \vee C)$ "

 apply (rule impI)
 apply (erule disjE)
 apply (erule disjE)
 apply (rule disjI1)
 apply assumption
 apply (rule disjI2)
 apply (rule disjI1)
 apply assumption
 apply (rule disjI2)
 apply (rule disjI2)
 apply assumption

done

lemma K: " $A \longrightarrow B \longrightarrow A$ "

 apply (rule impI)+
 apply assumption

done

lemma " $(A \vee A) = (A \wedge A)$ "

 apply (rule iffI)

 apply (erule disjE)
 apply (rule conjI)
 apply assumption+
 apply (rule conjI)
 apply assumption+

 apply (erule conjE)
 apply (rule disjI1)
 apply assumption

done

lemma S: " $(A \longrightarrow B \longrightarrow C) \longrightarrow (A \longrightarrow B) \longrightarrow A \longrightarrow C$ "

```

    apply (rule impI)+
    apply (erule impE)
    apply assumption
    apply (erule impE)
    apply assumption
    apply (erule impE)
    apply assumption+
done

```

```

lemma "(A  $\longrightarrow$  B)  $\longrightarrow$  (B  $\longrightarrow$  C)  $\longrightarrow$  A  $\longrightarrow$  C"
  apply (rule impI)+
  apply (erule impE)
  apply assumption
  apply (erule impE)
  apply assumption+
done

```

```

lemma " $\neg \neg$  A  $\longrightarrow$  A"
  apply (rule impI)
  apply (rule classical)
  apply (erule notE)
  apply assumption
done

```

```

lemma "A  $\longrightarrow$   $\neg \neg$  A"
  apply (rule impI)
  apply (rule notI)
  apply (erule notE)
  apply assumption
done

```

```

lemma " $(\neg$  A  $\longrightarrow$  B)  $\longrightarrow$  ( $\neg$  B  $\longrightarrow$  A)"
  apply (rule impI)+
  apply (rule classical)
  apply (erule impE)
  apply assumption
  apply (erule notE)
  apply assumption
done

```

```

lemma " $(($ A  $\longrightarrow$  B)  $\longrightarrow$  A)  $\longrightarrow$  A"
  apply (rule impI)+
  apply (rule classical)

```

```
    apply (erule impE)
    apply (rule impI)
    apply (erule notE, assumption)+
done
```

```
lemma "A  $\vee$   $\neg$  A"
  apply (rule classical)
  apply (rule disjI2)
  apply (rule notI)
  apply (erule notE)
  apply (rule disjI1)
  apply assumption
done
```

```
lemma " $(\neg (A \wedge B)) = (\neg A \vee \neg B)$ "
  apply (rule iffI)
```

```
    apply (rule classical)
    apply (erule notE)
    apply (rule conjI)
    apply (rule classical)
    apply (erule notE)
    apply (rule disjI1)
    apply assumption
    apply (rule classical)
    apply (erule notE)
    apply (rule disjI2)
    apply assumption

    apply (rule notI)
    apply (erule conjE)
    apply (erule disjE)
    apply (erule notE, assumption)+
done
```