A Graded Approach to Directions between Extended Objects

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Abstract. In this paper, we present the concept of *graded sections*, which allows us to compare alternative conceptualizations of direction relations and to process them in an integrative manner. To describe graded sections, *section bundles* are introduced, which provide formal means to (1) compare alternative candidates that are related via a direction relation like "north" or "south-east," (2) distinguish between good and not so good candidates, and (3) select a best candidate. The concepts and methods are exemplified with the cardinal direction "north," however, they are applicable to all cardinal directions, including the "cyclical east–west" and other directional terms, such as "left" and "in front of."

1 Introduction

Cardinal directions are widely used to specify spatial relations between geographic objects. Natural language terms, such as "north" or "south-east," play a core role in communication about space, both in human-human communication (e.g. for giving route descriptions) and in human-computer interaction (e.g., for querying geographic information systems). In this paper, we present qualitative characterizations of cardinal directions between extended objects. We show how different objects can be ranked and how well they fit a description involving a cardinal direction term. Possible applications of our approach are systems—like mobile devices—assisting a user via route descriptions or interfaces that support natural-language queries to a geographic information system, such as "give me all lakes south-west of Berlin."

A crucial step in processing cardinal directions based on such descriptions is the identification of an object, the *located object*, with respect to another object, the *reference object*. For example, the description "the cafeteria is north of the library" supplies information how to identify the cafeteria with respect to the library if the library is already identified. Such descriptions can be used in route instructions to assist a user or a cognitive agent navigating in an unknown environment. Given the spatial layout depicted in Fig. 1 several objects can be considered as being north of the library. In this sense, the description provides incomplete information. Since some objects are better candidates than others, the solution of the identification task involves

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the subtask of generating a sorted list of candidates that fulfill the spatial relation. In the current example, this leads to a sorted list of objects being north of the library, whose most prominent member is the best candidate for the cafeteria. Correspondingly, processing the query with the constraint "lakes south-west of Berlin" involves the task of generating a sorted list of lakes, from which the best candidates can be selected.

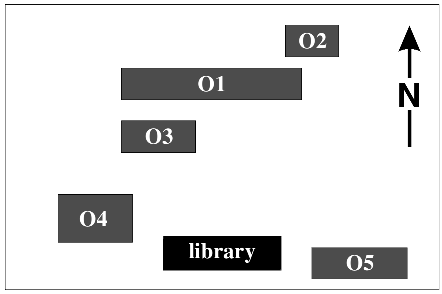


Fig. 1. Several objects located around a library

The reference object in the configuration depicted in Fig. 1 is the library. To identify the cafeteria the following question has to be answered: Which of the other five objects can be considered as being north of the library? Obviously, object O5 can be excluded. On the other hand, object O1 is an excellent candidate, because every longitude crossing the library also crosses O1. The objects O2 and O3 are north of the library as well, but object O1 seems to be a better candidate. In a different scenario, however, without object O1, the objects O2 and O3 would be quite good candidates. Under some interpretations even object O4 can be regarded as being north of the library.

Approaches modeling direction relations by a simple partition around a reference object (Section 1.2) only provide a binary decision whether a localized object, represented by a point, is north of the reference object or not. These approaches are not able to provide an order of different localized objects. There are at least two reasons why localized objects can be ordered regarding a direction relation: (1) the localized objects are extended and thus can lie in different sectors of the reference object, and (2) there are different alternatives for the sectors around the reference object. In this paper, we present an approach that can cope with both types of orders. The alternative sectors can be understood as graded sectors. They are described in the geometric formalization via a sector bundle.

We exemplify the idea of graded applicability in Fig. 2.A, which depicts four north-sectors of the library (for the geometrical analysis, see Section 2). To determine the direction relation between a located object and a reference object three alternatives are considered: crossing, overlapping, and inclusion. O1 crosses the dark grey sector, but the remaining objects do not cross any sector. Thus, regarding the crossing relation, O1 can be seen as the best candidate represented by the sorted list, $\{O1, O2\ O3\ O4\ O5\}_{CR}$. An object further left in the list is a better candidate and a missing comma indicates that two objects are equally good candidates. Using overlapping, O1 and O3 are equally good candidates, because they overlap all sectors. Furthermore, O3 is a better candidate than O2, since O2 does not overlap the dark grey sector. This leads to a second sorted list: $\{O1\ O3, O2, O4, O5\}_{OL}$. Taking inclusion as the underlying relation provides a third sorted list: $\{O1\ O2, O3, O4\ O5\}_{INC}$. Thus, O1 has a prominent status, because it is—independent of the underlying relation between the object region and the sectors—always one of the best candidates.

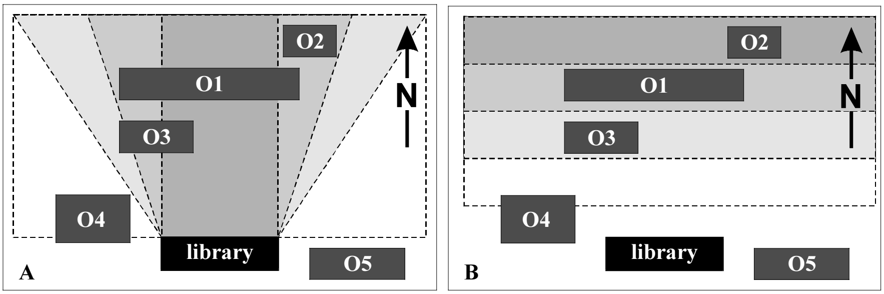


Fig. 2. Two alternatives of four possible sectors of the north-region of a library

The same procedure can be applied to determine sorted lists of candidates if we use systems of distance-based north-regions as depicted in Fig. 2.B. O2 can be considered further north than the other objects, because it is included in more north-regions than the other others.

Ranking objects based on a direction relation and a reference object depends on different conceptual parameters: First, the reference object can provide more than one north-sector or north-region. In the following, we use the term "section" for both types of spatial entities. Second, the three relations between the object region and the direction-based sections—overlap, inclusion, and crossing—can lead to different readings. Different settings of these conceptual parameters correspond to different conceptualizations of "being north of."

The geometry of section bundles for localizing extended objects is presented in Section 2. Kulik (2001) describes graded regions as the basis for relating points. The main focus of this paper lies on a formal characterization that emphasizes the common features of different ways of interpreting direction terms, and on systematic relations between these conceptualizations. We employ different spatial relations, such as crossing, inclusion, and overlapping, to determine the relative degree of membership of two objects to a graded section. These relative assessments are based on order relations specified in the next section.

A relative assessment for different candidates is a crucial step in solving an identification task. The focus in the example on the direction "north" is arbitrary. The considerations are valid for any kind of graded sections. Moreover, identification tasks are not confined to cardinal directions, but occur for other spatial relations, such as "in front of" and "left." The geometry presented in the following is not restricted to cardinal directions, but can be applied to other types of reference systems that follow the principles of graded localization. Thus, the approach can be used for instructed navigation of a cognitive agent (Tschander *et al.*, 2002).

1.1 Order Relations

The comparison of two objects with respect to such a direction-based localization as "north of the library" can be described by a pre-order. This section deals with the properties of pre-orders. A *pre-order* is a reflexive and transitive binary relation.

Given any binary relation Π the relation $\geq_\Pi$ as defined below is a pre-order since $\Rightarrow$ is a pre-order. *X has at least the same* Π*-degree as Y* ($\geq_\Pi$) iff Π relates X to every entity Z Π relates Y to. We use this general scheme to rank objects based on collections of sections.

$$\geq_\Pi(X, Y) \quad \Leftrightarrow_{\text{def}} \forall Z\, [\Pi(Y, Z) \Rightarrow \Pi(X, Z)]$$

The first and second argument of Π can have arguments of different sorts. A pre-order generates a family of order relations: its converse, denoted by $\leq_\Pi$, and the strict variants, denoted by $>_\Pi$ and $<_\Pi$. The strict variants are asymmetric.

$$\leq_\Pi(X, Y) \quad \Leftrightarrow_{\text{def}} \geq_\Pi(Y, X)$$

$$>_\Pi(X, Y) \quad \Leftrightarrow_{\text{def}} \geq_\Pi(X, Y) \wedge \neg\leq_\Pi(X, Y)$$

If two entities are related by $\geq_\Pi$ and $\leq_\Pi$, they *are equivalent regarding* Π ($\cong_\Pi$). Two entities *are incomparable regarding* Π ($\sim_\Pi$) if they cannot be related using the relations $\geq_\Pi$ or $\leq_\Pi$.

$$\cong_\Pi(X, Y) \quad \Leftrightarrow_{\text{def}} \geq_\Pi(X, Y) \wedge \leq_\Pi(X, Y)$$

$$\sim_\Pi(X, Y) \quad \Leftrightarrow_{\text{def}} \neg(\geq_\Pi(X, Y) \vee \leq_\Pi(X, Y))$$

We call a pre-order *linear* (or *exhaustive*) if any two geometric entities can be compared regarding the order.

$$\forall X\, Y \quad [\geq_\Pi(X, Y) \vee \leq_\Pi(X, Y)]$$

There are basically two strategies to represent different degrees of membership for a graded section: (1) a relative one and (2) an absolute one. An approach based on pre-orders to describe the ranking of objects to a graded section is a relative representation. This strategy is pursued in this paper. An absolute representation, such as fuzzy set theory, uses fixed values, typically between 0 and 1, to represent the degree of the membership of a point to a graded section. Fuzzy theories are frequently employed for geographic information systems (Schneider, 1999; Fisher, 2000). To determine the absolute degree of membership of a region to a graded section, we have to assign a weighted mean value for the whole region or have to consider the membership values of each individual point of the region. A general approach to represent the different degrees of spatial relationships between sharply bounded objects is proposed by Varzirgiannis (2000). His approach employs fuzzy logic to cover topological, directional, as well as distance relations.

1.2 Current Approaches to Cardinal Directions

Frank (1992) gives an overview and a classification of qualitative approaches applied to cardinal directions in geographic space. A general overview about qualitative spatial reasoning, especially for the field of Artificial Intelligence, can be found in Cohn (1997). Our approach focuses on qualitative descriptions of cardinal directions.

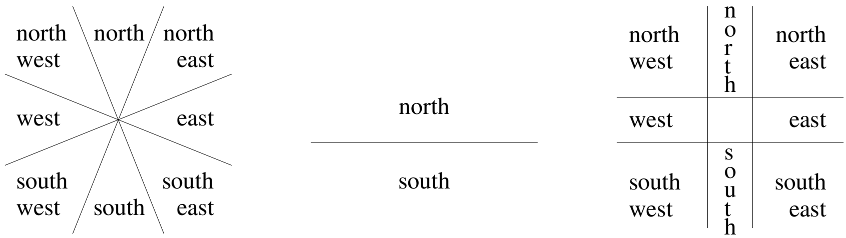


Fig. 3. The left figure illustrates a cone-based model of cardinal directions, the middle figure a projection-based model, and the right figure a projection-based model including a neutral zone

Formal approaches to cardinal directions can roughly be classified into two groups depending whether the spatial location of the located object is characterized by a point or a region. Point-based approaches are given in Peuquet and Zhan (1987), Frank (1996), Ligozat (1998), Kulik and Klippel (1999). Frank (1996) distinguishes cone-based and projection-based approaches (Fig. 3) to represent possible locations of a located object. In a cone-based model the region around the reference object is partitioned into four or eight cones. The projection-based model uses half-planes to divide the plane. A projection-based model can provide a neutral zone for extended reference objects (Fig. 3).

Approaches that take the extension of the located object into account employ different strategies to characterize the spatial relations based on cardinal directions. Papadias *et al.* (1995) associate minimum bounding rectangles to objects and compare the corresponding rectangles to relate the objects via directions. Escrig and Toledo (1998) adapt the model of Frank (1996) in the field of qualitative navigation. Goyal (2000) introduces—similar to Frank (1996)—a partition of the plane into nine sectors for a reference object. A *direction matrix* lists by the numbers 0 and 1, which of the nine sectors intersects the region of a located object. Depending on the degree of overlapping it is possible to assign values between 0 and 1 (Goyal and Egenhofer, 2001). Skiadopoulos and Koubarakis (2001) take up the work of Goyal and Egenhofer and investigate inferences resulting from the composition of cardinal directions. The approach of Schmidtke (2001) proposes a variety of geometric constructions that generate sectors based on a reference direction and an extended reference object. Schmidtke introduces the relation *crossing* between located objects and sectors to describe the concept of balance in a qualitative way.

The various cone- and projection-based models specifying cardinal directions show that there are different candidates for the north-sector or north-region of a reference object. Instead of searching for "the right one," we consider the whole family of possible sectors and regions as a basis to specify the different degrees of being north of a reference object. As geometric notion for systems of sectors and regions we introduce the "section bundle."

2 Geometry of Section Bundles

A graded section cannot be described by a single sharply bounded section. Thus, in the geometric characterization a graded section is specified by several sections. The geometric model of a graded section is the section bundle. The characterization does not assume that regions or sectors are sets of points. Therefore, the approach can be employed by every theory that supplies a characterization of regions (and sectors). However, we assume regular path-connected regions. In particular, theories of qualitative spatial reasoning based on mereotopology, like the RCC calculus of Randell *et al.* (1992), are compatible with our approach. A general overview about the correspondences between topological and order relations is given in Kainz (1990).

The formal characterization employs concepts and relations that have an obvious geometric interpretation and can be easily embedded in a geometry as specified by Hilbert (1956). The basic entities of the geometry are points, regions, sectors, sections, and section bundles. The capital P denotes points, R and R' denote regions, S and S' sectors, s and s' sections, and σ as well as σ' section bundles. The axiomatic characterization describes the properties of the geometric entities by axioms. The entities are related by the relation of incidence (ι). Further relations are defined below.

The goal of the geometry is a relative characterization of the different degrees of membership for regions to a section bundle. Therefore, we do not specify the geometry for sectors (Schmidtke, 2001).

2.1 Spatial Relations for Sections

A section is either a region or sector. A region either represents the spatial extension of a geographic object or is one of the regions of a section bundle. A sector is one of the "cones" of a section bundle. *A section is part of another section* ($\subseteq$) if every point of the former section is a point of the latter section (D1). *A region is included in a section* (i) if it is part of the section (D2). *A region overlaps a section* (o) if there is a point of the region that is a point of the section (D3). Since we have not specified a geometry of sectors, we provide an informal description of *crossing*, denoted by c, and note that the region of object O1 in Fig. 1 crosses the dark grey sector (for details, cf. Schmidtke, 2001). If a region crosses a section or is included in a section we write ci.

(D1) $s' \subseteq s \quad \Leftrightarrow_{\text{def}} \forall P\, [P\, \iota\, s' \Rightarrow P\, \iota\, s]$

(D2) $\mathsf{i}(R, s) \quad \Leftrightarrow_{\text{def}} R \subseteq s$

(D3) $\mathsf{o}(R, s) \quad \Leftrightarrow_{\text{def}} \exists P\, [P\, \iota\, R \wedge P\, \iota\, s]$

Due to the missing definition of crossing we assume that a region crossing a sector also overlaps it as an axiom (SC1).

(SC1) $\forall s\, R \quad [\mathsf{c}(R, S) \Rightarrow \mathsf{o}(R, S)]$

2.2 An Axiomatic Description of Section Bundles

According to axiom (SI1), a section bundle is uniquely determined by the sections that are incident with it. The sections are either regions or sectors, but a single section bundle is not specified by both types of geometric entities.

(SI1) $\forall \sigma\, \sigma' \quad [\forall s\, [s\, \iota\, \sigma \Leftrightarrow s\, \iota\, \sigma'] \Rightarrow \sigma = \sigma']$

We introduce an order relation for section bundles to compare the sections of a section bundle. The order relation is used in Section 2.3 to describe a relative characterization of two regions to a section bundle. A section s is called *at least as restrictive as* the section s' regarding the section bundle σ if both sections are incident with the section bundle and the section s is part of the section s' (D4). In this case we write $\geq(\sigma, s, s')$.

(D4) $\geq(\sigma, s, s')$ $\Leftrightarrow_{\text{def}}$ $s\,\iota\,\sigma \wedge s'\,\iota\,\sigma \wedge s \subseteq s'$

The objective is a characterization of section bundles that enables a relative description of the membership of two object regions to a section bundle. The axiom (SB2) ensures that two sections of a section bundle can always be compared regarding the relation $\geq$ since one section is at least as restrictive as the other one. Section 2.3 shows how we can use this axiom to introduce relative (non-numerical) degrees of membership of regions to a section bundle.

(SB2) $\forall \sigma\, s\, s'$ $[s\,\iota\,\sigma \wedge s'\,\iota\,\sigma \Rightarrow \geq(\sigma, s, s') \vee \geq(\sigma, s', s)]$

We mention some immediate consequences for regions and section bundles. The relation $\geq$ is reflexive, transitive, and antisymmetric for a given sector bundle. Theorems (T1) and (T2) state that if a region overlaps (is included in) a section s then every section that is less restrictive than the section s overlaps (includes) the region, too. In contrast, if a region crosses a sector then it crosses every sector that is at least as restrictive as the sector (SC2). Since we did not include the definition of c, we include this statement as an axiom.

(T1) $\forall \sigma\, s\, R$ $[\mathsf{o}(R, s) \Rightarrow \forall s'\, [\geq(\sigma, s, s') \Rightarrow \mathsf{o}(R, s')]]$

(T2) $\forall \sigma\, s\, R$ $[\mathsf{i}(R, s) \Rightarrow \forall s'\, [\geq(\sigma, s, s') \Rightarrow \mathsf{i}(R, s')]]$

(SC2) $\forall \sigma\, S\, R$ $[\mathsf{c}(R, S) \Rightarrow \forall S'\, [\geq(\sigma, S', S) \Rightarrow \mathsf{c}(R, S')]]$

If a section bundle has a section that is contained in all of its sections then this section is called the *core of the section bundle* (D5). The *hull of a section bundle* is a section that contains every section of the section bundle (D6).

(D5) $s = \mathsf{core}(\sigma)$ $\Leftrightarrow_{\text{def}}$ $s\,\iota\,\sigma \wedge \forall s'\, [s'\,\iota\,\sigma \Rightarrow \geq(\sigma, s, s')]$

(D6) $s = \mathsf{hull}(\sigma)$ $\Leftrightarrow_{\text{def}}$ $s\,\iota\,\sigma \wedge \forall s'\, [s'\,\iota\,\sigma \Rightarrow \geq(\sigma, s', s)]$

Employing (T1), (T2), and the axiom (SC2), we obtain the following results for the core and the hull of a section bundle: If a region overlaps (is included in) the core of a section bundle then it overlaps (is included in) every section of the section bundle. If a region crosses the hull of a section bundle then it crosses every sector of the section bundle. If a region does not overlap the hull of a section bundle, it does not overlap any of its sections.

2.3 Linear Orders Based on Overlapping, Inclusion, and Crossing

Depending on the underlying spatial relation there are different options to relate two regions regarding their relative degree of membership to a section bundle. The relative characterizations of degrees of membership are pre-orders (Section 1.1). We present five alternatives for pre-orders, which are based on overlapping, inclusion, and crossing.

Pre-orders are binary relations (Section 1.1). Formally, the description of the relative degree of membership of two regions to a section bundle is a ternary relation. The additional argument serves to restrict the quantification according to the form (D7). *A region R overlaps a section bundle at least to the same degree as a region R'* ($\geq_o$) if *R* overlaps every section of the section bundle which *R'* overlaps (D7).

(D7) $\geq_o(\sigma, R, R') \quad \Leftrightarrow_{def} \forall s\, [s\, \iota\, \sigma \Rightarrow (o(R', s) \Rightarrow o(R, s))]$

Since we consider the relative comparisons only for a fixed section bundle σ and do not compare different sector bundles, we can treat the relation $\geq_o(\sigma, R, R')$ for a fixed section bundle as a binary relation. Using the relations i, c, and ci, we obtain the corresponding relations $\geq_i$, $\geq_c$, and $\geq_{ci}$. The relation crossing is defined for sectors but not for regions. Thus, if the section bundle is determined by regions, the relation $\geq_{ci}$ is the same as $\geq_i$. *A region R belongs at least to the same degree as a region R'* to *a section bundle* ($\geq$) if both $\geq_{ci}$ and $\geq_o$ applies to *R* and *R'* (D8).

(D8) $\geq(\sigma, R, R') \quad \Leftrightarrow_{def} \geq_{ci}(\sigma, R, R') \wedge \geq_o(\sigma, R, R')$

For a fixed section bundle all four relations $\geq_o$, $\geq_i$, $\geq_c$, and $\geq_{ci}$ are binary relations. Since $\geq$ is a conjunction of the pre-orders $\geq_{ci}$ and $\geq_o$ for a fixed section bundle σ, it is also pre-order for σ. The definitions for $\leq_o$, $\cong_o$, $\leq_i$, etc. are as expected (Section 1.1). *A region R overlaps a section bundle more than a region R'* ($>_o$) if *R* overlaps every section of the section bundle which *R'* overlaps but *R'* does not overlap every section which *R* overlaps. The definitions for $>_i$, …, $<_o$, $<_i$, … apply accordingly.

A region *R* overlaps a section bundle more than a region *R'* if and only if there is a section of the section bundle overlapping *R* but not *R'* (T3). From this follows the linearity of the strict order $>_o$ (T4). We can prove results analogous to (T3) and (T4) also for the strict orders $>_i$ and $>_c$, but not for the relation $>_{ci}$ and $>$.

(T3) $\forall \sigma\, R\, R' \quad [>_o(\sigma, R, R') \Leftrightarrow \exists s\, [s\, \iota\, \sigma \wedge o(R, s) \wedge \neg o(R', s)]]$

(T4) $\forall \sigma\, R\, R' \quad [>_o(\sigma, R, R') \vee \cong_o(\sigma, R, R') \vee <_o(\sigma, R, R')]$

Proof of (T3): We prove the direction "⇐". Let *s* be a section of the section bundle σ fulfilling $o(R, s)$ and $\neg o(R', s)$ for the regions *R* and *R'*. Applying (SB2) we get for every $s'\, \iota\, \sigma$: $s \subseteq s' \vee s' \subseteq s$. Hence, for every *s'* we obtain: If $s \subseteq s'$ holds then we get according to (T1) $o(R, s')$ and if $s' \subseteq s$ holds then $\neg o(R', s')$. Thus we get $\geq_o(\sigma, R, R')$ and because of $\neg o(R', s)$ we obtain $>_o(\sigma, R, R')$. ♦

The relation $\geq_o$ is a linear pre-order. Of two regions, one region overlaps a section bundle either at least or at most to the same degree as the other region (T5). The proof of (T5) is based on (T4). We can prove the linearity for $\geq_i$ and $\geq_c$ as well.

(T5) $\forall \sigma\, R\, R' \quad [\geq_o(\sigma, R, R') \vee \leq_o(\sigma, R, R')]$

We cannot prove the linearity for the relations $\geq_{ci}$ and $\geq$ since both of them allow cases where two regions are incomparable. The objects O2 and O3 in Fig. 2.A, for instance, are incomparable regarding the relation $\geq$, because O2 is included in the sector that does not include O3 whereas O3 overlaps the core sector.

2.4 Verification Conditions

The relation $\geq$ ($\geq_{ci}$) distinguishes more spatial configurations than the three (two) relations $\geq_o$, $\geq_i$, and $\geq_c$ ($\geq_i$ and $\geq_c$). On the other hand, for the strict variants of the three relations it suffices to find a single section to determine the relative degree of membership to a section bundle (T3). The relations $\geq$ and $\geq_{ci}$ are generally more difficult to determine since they are not linear. We present conditions that simplify—similar to (T3)—the determination of the relations $\geq$ and $\geq_{ci}$ on certain assumptions. They follow from (T1), (T2), and the axiom (SC2). If a region does not overlap the hull of a section bundle then every other region belongs to the section bundle at least to the same degree (T6). A region that is included in the core of a section bundle belongs at least to the same degree to the section bundle as every other region (T7). If a region crosses the hull of a section bundle it crosses every other section of the section bundle and thus belongs at least to the same degree to the section bundle as every other region (T8). Analogous results like (T7) and (T8) hold for the relation $\geq_{ci}$.

(T6) $\forall \sigma\, R \quad [\neg \mathsf{o}(R, \mathsf{hull}(\sigma)) \Rightarrow \forall R' [\geq(\sigma, R', R)]]$

(T7) $\forall \sigma\, R \quad [\mathsf{i}(R, \mathsf{core}(\sigma)) \Rightarrow \forall R' [\geq(\sigma, R, R')]]$

(T8) $\forall \sigma\, R \quad [\mathsf{c}(R, \mathsf{hull}(\sigma)) \Rightarrow \forall R' [\geq(\sigma, R, R')]]$

3 Section Bundles and Cardinal Directions

We restrict the analysis of cardinal directions between extended objects to maps and sections of the world, such the visible environment of a cognitive agent, that do not include the North or South Pole. In general, the relations “further north/further south” constitute total linear bounded orders, whereas the relations “further west/further east” are cyclic orders. Nevertheless, for maps and sections of the world excluding the poles the relations “further west/further east” are also total linear bounded orders and all relations concerning cardinal directions can be described formally in the same way (Kulik and Klippel, 1999).

To describe the cardinal direction between a reference object and a localized object we assign to a reference object a section bundle for each cardinal direction. A localized object is north of the reference object if—depending on the interpretation—the region of the located object overlaps, is included, or crosses the hull of the section bundle.

There are two alternatives to relate extended objects regarding cardinal directions. They can be described in the case of the north direction as “further north” vs. “more to the north” (Section 1 and Fig. 2). We denote the spatial extension of an object O by $\mathsf{Loc}(O)$. In the first case, the north-boundary of a map or an underlying section of the world induces a section bundle with different sections (denoted by sb): An object O is *further north than* an object O' if the object region $\mathsf{Loc}(O)$ belongs more to the section bundle of the north-boundary than the object region $\mathsf{Loc}(O')$ (D9).

(D9) $\mathsf{north_f}(O, O') \quad \Leftrightarrow_{def} \mathsf{sb}(\sigma, \text{north-boundary}) \wedge \geq(\sigma, \mathsf{Loc}(O), \mathsf{Loc}(O'))$

For a map like in Fig. 2.B that is aligned to the North, the upper boundary of the map induces a section bundle of the north-direction and the other map boundaries accordingly for the other cardinal directions.

In the second case, we associate for the north direction a section bundle σ to the reference object *O* (denoted by nsb(σ, *O*)): An object *O'* is *more to the north than* an object *O"* if the object region *O'* belongs more to the section bundle of the reference object than the object region *O"* (D10).

(D10) $\mathrm{north}_m(O, O', O'') \Leftrightarrow_{\mathrm{def}} \mathrm{nsb}(\sigma, O) \wedge >(\sigma, \mathrm{Loc}(O'), \mathrm{Loc}(O''))$

The cone-based model (Fig. 3) can be interpreted as a special case of a section bundle. The cone denoting the north direction can be considered as the core of the section bundle and the hull of the section bundle consists of the sector determined by the points of three cones denoting the north-west, north, and north-east direction. Another example is given in Fig. 2.A.

In lieu of relation > we can also employ the relations $>_o$, $>_i$, $>_c$, and $>_{ci}$ in (D10). The formal specification does not specify which of the orders is the right one. The decision, which order has to be taken, depends on the particular application and scenario. However, the possibility to integrate each of the orders without a commitment to a specific numerical representation shows the flexibility of our approach. In case of the library-example, we obtain for the object regions given the sectors in Fig. 2.A (we use the infix notation for the relations and omit the sector bundle): O1 > O2 ~ O3 > O4 > O5, O1 $>_c$ O2 $\cong_c$ O3 $\cong_c$ O4 $\cong_c$ O5, O1 $\cong_o$ O3 $>_o$ O2 $>_o$ O4 $>_o$ O5, O1 $\cong_i$ O2 $>_i$ O3 $>_i$ O4 $\cong_i$ O5, O1 $>_{ci}$ O2 $>_{ci}$ O3 $>_{ci}$ O4 $\cong_{ci}$ O5. The results for the relations $>_c$, $>_o$, and $>_i$ agree with the sorted lists $\{\}_{CR}$, $\{\}_{OL}$, and $\{\}_{INC}$ of the introduction.

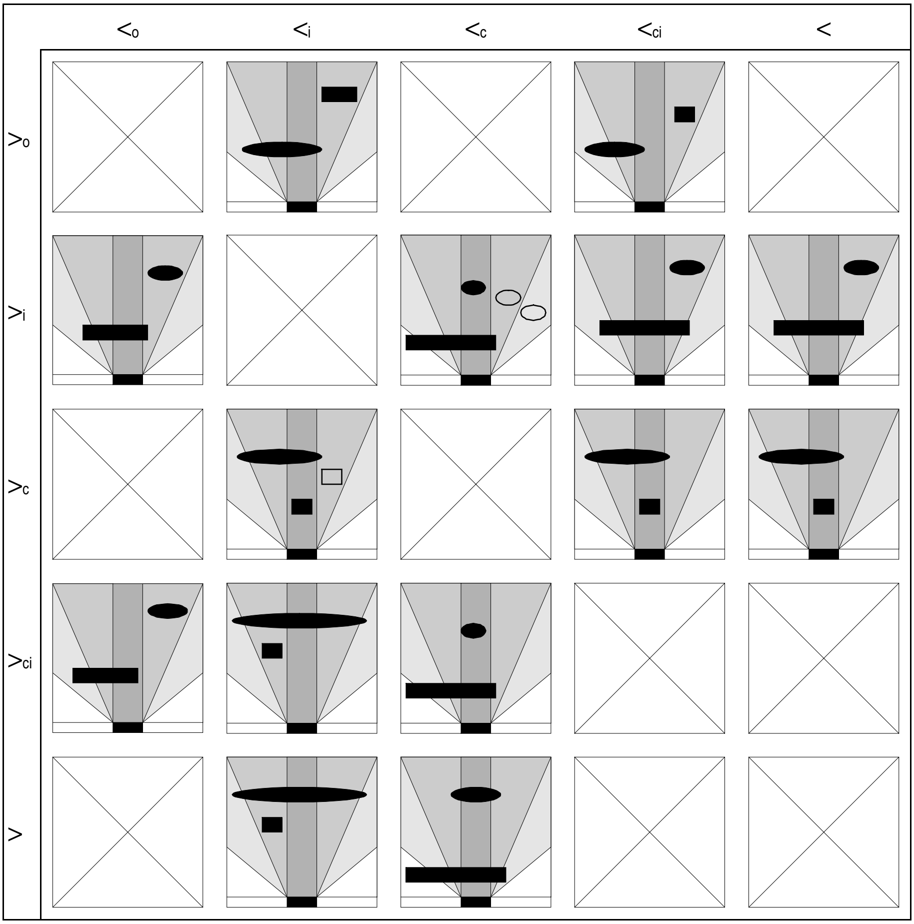
In Table 1, we give an overview of situations that are evaluated differently by distinct choices. This overview can be used to determine which spatial relation (i.e., overlapping, inclusion, or crossing) should be chosen if a certain interpretation for two objects regarding the north-direction is desired. Table 1 is based on the following three rules: (1) Every strict linear pre-order is asymmetric, (2) the relation > is a conjunction of two pre-orders, for instance, the relation $>_o$ implies ¬<, and (3) the relation $>_c$ ($>_o$) implies $\neg<_o$ ($\neg<_c$). The table should be read as follows: The pre-orders noted in the left column and in the upper row refer to the relations between the oval and the rectangle. For example, the entry in row 1 and column 2 means that the relations $>_o$ and $<_i$ hold for the oval and the rectangle. In case of the north direction, the comparison based on overlapping leads to the interpretation that the oval is more to the north than the rectangle, whereas the comparison based on inclusion suggests that the rectangle is more to the north than the oval.

4 Conclusions

The presented geometry introduces section bundles to enable a relative characterization of directional concepts, such as "north" or "in front of." Section bundles describe graded sections that allow us to assess which of two extended objects belongs more to a graded section. Thus, we can compare alternative candidates that are related via a direction concept such as "north" and rank candidates to solve the identification task described in the introduction.

Since we consider direction relations between extended objects, different applications may require different alternatives to order the objects for a given section bundle. Our approach shows different ways to introduce an order for the located objects regarding a direction relation. They depend on whether the overlapping, the inclusion, or the crossing of the object regions with the sections of a section bundle of a reference object is considered.

Table 1. Spatial constellations with conflicting judgments regarding different pre-orders (the outlined regions show further alternatives)



In case of the north-direction, the approach can be used to model the interpretation that an extended object is more to the north than another extended object regarding a fixed reference object, and it can be employed to describe the interpretation that an object is further north than another object. Both interpretations can be characterized in a uniform geometric way without referring to analytical geometry or using numerical concepts.

The approach used in this paper is a relative characterization of graded membership to determine from two regions the one that belongs more to a graded section (Section 1.1). For a relative assessment a numerical representation is not required. To determine the relative degree of membership of a region to a section bundle, we have to evaluate—depending on the underlying spatial relation of regions and sections—which sections of a section bundle overlap, include, or cross a region. Provided that the regions are comparable regarding their membership to a section bundle, the assumption suffices that the degrees of membership can be described by a (linear) pre-order.

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