Advances in Programming Languages

APL5: Hoare logic

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Topic: Some formal verification

The next three lectures will be about some techniques and tools for formal verification, specifically:

- Hoare logic
- JML: The Java Modeling Language
- ESC/Java 2: The Extended Static Checker for Java

Outline

1. Introduction
2. Axioms, meaning and truth
3. Applications
4. Summary

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First-order logic

A formal language for describing certain kinds of logical assertion.

Variables $x, y, z, x_1, \ldots$ Terms $e ::= x \mid f(e_1, \ldots, e_n)$

$$\begin{aligned}\text{Formulae} \quad P, Q ::= &\ \mathbf{true} \mid \mathbf{false} \mid R(e_1, \ldots, e_n) \\ \mid &\ P \wedge Q \mid P \vee Q \mid P \rightarrow Q \mid \neg P \\ \mid &\ \forall x.P \mid \exists x.P\end{aligned}$$

A *function* like $f(\ldots)$ has a fixed number of arguments, its *arity*. This might be zero, one or more. For example: 5, $sqrt(-)$, $+$.

A *predicate* like $R(\ldots)$ also has an arity: zero (a *proposition*), one (a *predicate*), or more (a *relation*). For example: $\mathbf{true}$, $Even(-)$, $<$, $=$, $Divides(-,-)$.

$$\forall x, y.(x > 5) \wedge (y > x) \rightarrow (x + y > 10)$$

A simple imperative language

Pick a minimal language of commands and variable assignment.

Variables a, b, i, n, . . . Expressions E, B ::= a | F(E1,...,En)

Code C ::= **skip** | a:=E | C;C
| **if** B **then** C **else** C | **while** B **do** C

Variables like a, b here are storage cells, distinct from logical variables x, y.

Functions F have an arity, and we assume useful ones like 0, 1, +, sqrt(−).

For example, the following computes the factorial of n and places it in variable m:

i := n; a := 1; **while** i>0 **do** (a := a∗i; i: = i−1); m := a

Hoare triples

A *Hoare triple* is an assertion about the behaviour of a program fragment.

$$\{P\}\ C\ \{Q\}$$

Here we have:

- An imperative program C.
- A *precondition* P and a *postcondition* Q: logical formulae concerning the state of the program variables.

The triple asserts that for any terminating run of the program, if P holds before then Q holds afterwards.

$$\{a > 3\}\ b := a{+}a\ \{b > 6\}$$

$$\{d > z \wedge d' > z\}\ d := d{*}d'\ \{d > z^2\}$$

$$\{\mathbf{true}\}\ \mathbf{while}\ i{>}0\ \mathbf{do}\ i := i{-}1\ \{i \leqslant 0\}$$

Partial vs. Total

Partial: $\{P\}$ C $\{Q\}$ does not assert that C will terminate when started in a state satisfying P, only that Q will hold if it does.

The alternative *total* triple $[P]$ C $[Q]$ does assert that C terminates, but in practice methods for proving termination are often quite different to methods for proving properties like Q.

Hypothetical: $\{P\}$ C $\{Q\}$ makes no claim that P actually will be true when C is executed, only what will happen if it is.

Imprecise: $\{P\}$ C $\{Q\}$ may not include all that can be deduced about C.

For example, $\{\mathbf{true}\}$ C $\{\mathbf{true}\}$ is always valid, but rarely useful.

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Hoare rules

Hoare set out a number of rules for how to deduce triples.

$$\frac{}{\{P\}\ \textbf{skip}\ \{P\}} \qquad \frac{\{P\}\ C\ \{Q\} \qquad \{Q\}\ C'\ \{R\}}{\{P\}\ C;C'\ \{R\}} \qquad \frac{}{\{P[E/x]\}\ x{:=}E\ \{P\}}$$

$$\frac{\{P \wedge (B = \mathrm{true})\}\ C\ \{Q\} \qquad \{P \wedge (B \neq \mathrm{true})\}\ C'\ \{Q\}}{\{P\}\ \textbf{if}\ B\ \textbf{then}\ C\ \textbf{else}\ C'\ \{Q\}}$$

$$\frac{\{P \wedge (B = \mathrm{true})\}\ C\ \{P\}}{\{P\}\ \textbf{while}\ B\ \textbf{do}\ C\ \{P \wedge (B \neq \mathrm{true})\}}$$

$$\frac{P \to P' \quad \{P'\}\ C\ \{Q'\} \quad Q' \to Q}{\{P\}\ C\ \{Q\}}$$

Rules have also been proposed for several other programming language features: concurrency, procedures, local variables, pointers,. . .

In fact, the last rule is not as strong as it might be, but this was not realised for several years. See for example [Nipkow CSL 2002 §3] for some of the history.

Truth and beauty

We write $\vdash \{P\}\ C\ \{Q\}$ when a triple can be derived using the rules. But is such a triple true? This depends on the meaning of C, its *semantics*. Which is what, exactly?

- Hoare proposed an *axiomatic semantics*: the derivable triples $\vdash \{P\}\ C\ \{Q\}$ are what define the meaning of C.
- An alternative is to define the behaviour of C separately, and write $\vDash \{P\}\ C\ \{Q\}$ when a triple holds true in this semantics.

There are various such ways to define the behaviour of C:

- *Operational semantics* specifies how one term executes to give another.
- *Denotational semantics* maps programs into a separate mathematical domain.
- An *abstract machine* evaluates code by executing the steps of some simplified processor.

In all cases we then want to compare $\vdash$ (derived) with $\vDash$ (observed).

Operational semantics

An operational semantics here must track commands C and program states S, where S(x) gives the value of variable x in state S.

- A *small-step* semantics $S,C \to S',C'$ reduces programs little by little:

$$S,(a:=5;C) \quad \longrightarrow \quad S'[a \leftarrow 5],C$$

- A *big-step* semantics $S,C \Downarrow S'$ evaluates programs to a final state:

$$S,(i:=5;j:=1;\textbf{while}\ i>0\ \textbf{do}\ (i:=i-1;j:=j*2)) \Downarrow S[i \leftarrow 0, j \leftarrow 32]$$

Either of these can themselves be defined by derivation rules, using the approach of *Structural Operational Semantics*. [Plotkin 1981]

Soundness and completeness

Given a semantics, we can identify which triples are valid:

$$\models \{P\}\ C\ \{Q\} \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \forall S,T\ .\ (P(S) \wedge S,C \Downarrow T) \rightarrow Q(T)$$

This gives a means to assess the derivation rules for triples:

Soundness Every derivable triple is valid:

$$\vdash \{P\}\ C\ \{Q\} \quad \Longrightarrow \quad \models \{P\}\ C\ \{Q\}$$

Completeness Every valid triple can be derived using the rules:

$$\models \{P\}\ C\ \{Q\} \quad \Longrightarrow \quad \vdash \{P\}\ C\ \{Q\}$$

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Reasoning and specification

Hoare logic supports quite general reasoning about imperative programs and their behaviour. However, the two most common applications are:

Specification Stating what properties a program ought to have, either by annotating existing code, or before any is written.

Verification Checking that a program does indeed have these desired properties.

In practice, this means generalising pre- and postconditions to include:

Assertions about the state at some point within a program

Loop invariants to hold at each repeat of a loop;

Object invariants that each method is to maintain;

Method constraints as pre- and postconditions on method invocation.

Hoare in verification tools

The general approach for Hoare-style formal verification tools is this:

- A programmer annotates source code, or a library interface.
- A tool analyses the code and attempts to show that all the assertions given can be derived using the standard rules.
- The tool may be able to do this unassisted.
- If not, it emits *verification conditions*, purely logical assertions that need to be checked.
- These may be passed on to an automated theorem prover, or some other *decision procedure*.

Design by Contract™

Design by Contract™ (DBC) is a software design methodology promoted by Bertrand Meyer and the *Eiffel* programming language.

DBC makes Hoare logic a vital component in program development, strengthening it to the notion of a *contract*:

- The precondition of a procedure imposes an obligation on any caller;
- In return, the procedure must guarantee that the specified postcondition will hold when it exits.

The contract also includes additional information such as side-effects, invariants, and error conditions.

NB: this modifies the *hypothetical* aspect of Hoare logic, where a precondition is “supposing”

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Summary

- Hoare logic triples $\{P\}\ C\ \{Q\}$ make logical assertions about imperative code.
- The *soundness* and *completeness* of Hoare reasoning can be tested with respect to a program's *semantics*.
- Hoare assertions are used in *specification* to annotate programs and libraries.
- Tools can carry out automated *verification* against these assertions.
- Design by contract™ strengthens these into *contracts*.

Resources

Not specifically for this lecture, but for general chitchat about current programming language issues:

- Lambda the Ultimate: Programming languages weblog.
  Some astonishing enthusiasm for heavy programming language theory.
- `http://developers.slashdot.org`
  One channel on the self-proclaimed *News for Nerds*. Occasional programming language issues, lots of comments but can be thin on content. Beware of the trolls.
- `comp.lang.<almost-any-language>`, `comp.lang.functional`
  Programming language newsgroups, some very busy. `c.l.f` has a endless supply of questioners, and some very patient responders.