Advances in Programming Languages

APL5: Hoare logic

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Semester 2 Week 4
The next three lectures will be about some techniques and tools for formal verification, specifically:

- Hoare logic
- JML: The Java Modeling Language
- ESC/Java 2: The Extended Static Checker for Java
Outline

1. Introduction
2. Axioms, meaning and truth
3. Applications
4. Summary
First-order logic

A formal language for describing certain kinds of logical assertion.

Variables $x, y, z, x_1, \ldots$  
Terms $e ::= x \mid f(e_1, \ldots, e_n)$  

Formulae $P, Q ::= true \mid false \mid R(e_1, \ldots, e_n) \mid P \land Q \mid P \lor Q \mid P \rightarrow Q \mid \neg P \mid \forall x.P \mid \exists x.P$

A function like $f(\ldots)$ has a fixed number of arguments, its arity. This might be zero, one or more. For example: 5, $\text{sqrt}(-)$, $\text{+}$.  

A predicate like $R(\ldots)$ also has an arity: zero (a proposition), one (a predicate), or more (a relation). For example: $true$, $\text{Even}(-)$, $<$, $=\text{Divides}(\text{--})$.  

$$\forall x, y. (x > 5) \land (y > x) \rightarrow (x + y > 10)$$
A simple imperative language

Pick a minimal language of commands and variable assignment.

Variables  a, b, i, n, . . .  Expressions  E, B ::= a | F(E1, . . . , En)

Code  C ::= skip | a := E | C; C

| if B then C else C | while B do C

Variables like a, b here are storage cells, distinct from logical variables x, y.

Functions F have an arity, and we assume useful ones like 0, 1, +, sqrt(−).

For example, the following computes the factorial of n and places it in variable m:

i := n; a := 1; while i > 0 do (a := a * i; i := i − 1); m := a
A *Hoare triple* is an assertion about the behaviour of a program fragment.

$$\{P\} \ C \ \{Q\}$$

Here we have:

- An imperative program $C$.
- A *precondition* $P$ and a *postcondition* $Q$: logical formulae concerning the state of the program variables.

The triple asserts that for any terminating run of the program, if $P$ holds before then $Q$ holds afterwards.

$$\{a > 3\} \ b := a+a \ \{b > 6\}$$
$$\{d > z \land d' > z\} \ d := d*d' \ \{d > z^2\}$$
$$\{\text{true}\} \ \textbf{while} \ i > 0 \ \textbf{do} \ i := i-1 \ \{i \leq 0\}$$
**Partial:** \{P\} C \{Q\} does not assert that C will terminate when started in a state satisfying P, only that Q will hold if it does.

The alternative *total* triple \[P\] C \[Q\] does assert that C terminates, but in practice methods for proving termination are often quite different to methods for proving properties like Q.

**Hypothetical:** \{P\} C \{Q\} makes no claim that P actually will be true when C is executed, only what will happen if it is.

**Imprecise:** \{P\} C \{Q\} may not include all that can be deduced about C.

For example, \{true\} C \{true\} is always valid, but rarely useful.
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Hoare rules

Hoare set out a number of rules for how to deduce triples.

\[
\begin{align*}
\{P\} \text{skip} \{P\} \\
\{P\} C \{Q\} \quad \{Q\} C' \{R\} \\
\{P\} C; C' \{R\} \\
\{P[E/x]\} \ x:=E \ \{P\}
\end{align*}
\]

\[
\begin{align*}
\{P \land (B = \text{true})\} \ C \ \{Q\} \\
\{P \land (B \neq \text{true})\} \ C' \ \{Q\}
\end{align*}
\]

\[
\{P\} \ \text{if} \ B \ \text{then} \ C \ \text{else} \ C' \ \{Q\}
\]

\[
\begin{align*}
\{P \land (B = \text{true})\} \ C \ \{P\} \\
\{P\} \ \text{while} \ B \ \text{do} \ C \ \{P \land (B \neq \text{true})\}
\end{align*}
\]

\[
\begin{align*}
P & \rightarrow P' \\
\{P'\} \ C \ \{Q'\} \\
\{P\} \ C \ \{Q\}
\end{align*}
\]

Rules have also been proposed for several other programming language features: concurrency, procedures, local variables, pointers, . . .

In fact, the last rule is not as strong as it might be, but this was not realised for several years. See for example [Nipkow CSL 2002 §3] for some of the history.
We write $\vdash \{P\} \ C \ \{Q\}$ when a triple can be derived using the rules. But is such a triple true? This depends on the meaning of $C$, its *semantics*. Which is what, exactly?

- Hoare proposed an *axiomatic semantics*: the derivable triples $\vdash \{P\} \ C \ \{Q\}$ are what define the meaning of $C$.
- An alternative is to define the behaviour of $C$ separately, and write $\models \{P\} \ C \ \{Q\}$ when a triple holds true in this semantics.

There are various such ways to define the behaviour of $C$:

- *Operational semantics* specifies how one term executes to give another.
- *Denotational semantics* maps programs into a separate mathematical domain.
- An *abstract machine* evaluates code by executing the steps of some simplified processor.

In all cases we then want to compare $\vdash$ (derived) with $\models$ (observed).
An operational semantics here must track commands $C$ and program states $S$, where $S(x)$ gives the value of variable $x$ in state $S$.

- A *small-step* semantics $S, C \rightarrow S', C'$ reduces programs little by little:

  $$S, (a:=5;C) \rightarrow S'[a \leftarrow 5], C$$

- A *big-step* semantics $S, C \Downarrow S'$ evaluates programs to a final state:

  $$S, (i:=5; j:=1; \textbf{while } i>0 \textbf{ do } (i:=i-1; j:=j*2)) \Downarrow S[i \leftarrow 0, j \leftarrow 32]$$

Either of these can themselves be defined by derivation rules, using the approach of *Structural Operational Semantics*. [Plotkin 1981]
Soundness and completeness

Given a semantics, we can identify which triples are valid:

$$\models \{P\} C \{Q\} \overset{\text{def}}{\iff} \forall S,T . (P(S) \land S,C \downarrow T) \rightarrow Q(T)$$

This gives a means to assess the derivation rules for triples:

**Soundness**  Every derivable triple is valid:

$$\vdash \{P\} C \{Q\} \implies \models \{P\} C \{Q\}$$

**Completeness**  Every valid triple can be derived using the rules:

$$\models \{P\} C \{Q\} \implies \vdash \{P\} C \{Q\}$$
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Reasoning and specification

Hoare logic supports quite general reasoning about imperative programs and their behaviour. However, the two most common applications are:

**Specification**  Stating what properties a program ought to have, either by annotating existing code, or before any is written.

**Verification**  Checking that a program does indeed have these desired properties.

In practice, this means generalising pre- and postconditions to include:

**Assertions**  about the state at some point within a program

**Loop invariants**  to hold at each repeat of a loop;

**Object invariants**  that each method is to maintain;

**Method constraints**  as pre- and postconditions on method invocation.
The general approach for Hoare-style formal verification tools is this:

- A programmer annotates source code, or a library interface.
- A tool analyses the code and attempts to show that all the assertions given can be derived using the standard rules.
- The tool may be able to do this unassisted.
- If not, it emits verification conditions, purely logical assertions that need to be checked.
- These may be passed on to an automated theorem prover, or some other decision procedure.
Design by Contract™ (DBC) is a software design methodology promoted by Bertrand Meyer and the *Eiffel* programming language.

DBC makes Hoare logic a vital component in program development, strengthening it to the notion of a *contract*:

- The precondition of a procedure imposes an *obligation* on any caller;
- In return, the procedure must *guarantee* that the specified postcondition will hold when it exits.

The contract also includes additional information such as side-effects, invariants, and error conditions.

NB: this modifies the *hypothetical* aspect of Hoare logic, where a precondition is “supposing”
Hoare logic triples $\{P\} C \{Q\}$ make logical assertions about imperative code.

The *soundness* and *completeness* of Hoare reasoning can be tested with respect to a program’s *semantics*.

Hoare assertions are used in *specification* to annotate programs and libraries.

Tools can carry out automated *verification* against these assertions.

Design by contract™ strengthens these into *contracts*. 
Not specifically for this lecture, but for general chitchat about current programming language issues:

- **Lambda the Ultimate: Programming languages weblog.** Some astonishing enthusiasm for heavy programming language theory.

- **http://developers.slashdot.org**
  One channel on the self-proclaimed *News for Nerds.* Occasional programming language issues, lots of comments but can be thin on content. Beware of the trolls.

- **comp.lang.<almost-any-language>, comp.lang.functional**
  Programming language newsgroups, some very busy. c.l.f has an endless supply of questioners, and some very patient responders.