AGTA Tutorial Sheet 4

Please read and attempt these questions before coming to the tutorial.

1. Use iterated elimination of strictly dominated strategies in order to find the unique Nash Equilibrium in this finite 2-player (bimatrix) game, by first reducing the game to a 2×2 game. (Recall that a pure strategy may be strictly dominated by a mixed strategy.)

$$\begin{bmatrix} (5,2) & (22,4) & (4,9) & (7,6) \\ (16,4) & (18,5) & (1,10) & (10,2) \\ (15,12) & (16,9) & (18,10) & (11,3) \\ (9,15) & (23,9) & (11,5) & (5,13) \end{bmatrix}$$

2. Recall that a finite n-player normal form game, G, is given by the payoff table that specifies, for each player $i \in \{1,\ldots,n\}$, and for each combination $(s_1,\ldots,s_n)$ of pure strategies for the n players the payoff $u_i(s_1,\ldots,s_n)$ for player i. (Here each $s_j \in S_j$ is a pure strategy for player $j = 1,\ldots,n$.)

   In this question we shall consider how to compute an important notion of equilibrium which generalizes Nash equilibrium, called "correlated equilibrium".

   For a finite n-player game, G, with pure strategy sets $S_1,\ldots,S_n$, a ***correlated distribution*** is a probability distribution, $p(\cdot)$, on the set of possible pure-strategy combinations (i.e., pure-strategy profiles), $(s_1,\ldots,s_n) \in S_1 \times \ldots \times S_n = S$. In other words, for all $(s_1,\ldots,s_n) \in S$, $p(s_1,\ldots,s_n) \geq 0$, and $\sum_{(s_1,\ldots,s_n)\in S} p(s_1,\ldots,s_n) = 1$.

   Note: a correlated distribution $p(\cdot)$ is *not necessarily* a distribution on combinations of pure strategies that arises from a mixed strategy profile $x = (x_1,\ldots,x_n)$, where the probability of each pure combination $(s_1,\ldots,s_n)$ is the product of the probabilities $\prod_{i=1}^n x_i(s_i)$ with which each player i randomly (independently) chooses its own pure strategy s_i according to distribution $x_i(\cdot)$. Such a joint distribution on pure-strategy profiles is called a *product distribution* on $S = S_1 \times \ldots \times S_n$, whereas a correlated distribution can be *any* distribution on S.

   Interpretation: imagine that there is a "randomized recommender", who uses the probability distribution $p(\cdot)$ to randomly select a pure-strategy profile $(s_1,\ldots,s_n)$, and who then suggests to each player i to play pure strategy s_i. Assume every player knows the distribution p.

   For a given correlated distribution p, for a given pure strategy $s_i \in S_i$ for player i, and for a given combination $s_{-i} = (s_1, s_2, \ldots, s_{i-1}, \texttt{empty}, s_{i+1}, \ldots, s_n) \in S_{-i}$ of pure strategies for all the other players (other than player i), let

$$p(s_{-i} \mid s_i) = \frac{p(s_{-i}; s_i)}{\sum_{t_{-i}\in S_i} p(t_{-i}; s_i)}$$

   denote the conditional probability of the combination of strategies $s = (s_1,\ldots,s_n)$, given that the pure strategy s_i was recommended to player i by the "random recommender" (who uses joint distribution p to pick everyone's recommendation).

Now suppose player i believes that other players are inclinded to follow the recommendation of the recommender. If player i is recommended pure strategy $s_i \in S_i$, and chooses to play pure strategy $s'_i \in S_i$, let

$$U_i^{s'_i}(p \mid s_i) := \sum_{s_{-i} \in S_{-i}} p(s_{-i} \mid s_i) \cdot u_i(s_{-i}; s'_i)$$

denote the *conditional expected payoff* to player i, for playing pure strategy s'_i, assuming *other* players play according to their recommendations from correlated distribution p on pure-strategy profiles, *conditioned* on the fact that player i itself was recommended the pure strategy s_i.

Definition: A correlated distribution $p(\cdot)$ is a ***correlated equilibrium*** (CE) for the game if for all players i and all pure strategies $s_i, s'_i \in S_i$,

$$U_i^{s_i}(p \mid s_i) \geq U_i^{s'_i}(p \mid s_i)$$

In other words, no single player would be strictly better off by unilaterally deviating from its own recommendation s_i under the distribution p (its expected payoff, conditioned on the pure strategy s_i that it was recommended, would not be increased by deviating) using a different strategy s'_i.

Questions:

(a) Explain why every Nash equilibrium $x = (x_1, \ldots, x_n)$ necessarily also gives a CE, defined by the correlated distribution $p(\cdot)$ given by the product distribution $p(s_1, \ldots, s_n) = \prod_{i=1}^{n} x_i(s_i)$. Thus, conclude that every finite game does have at least one CE.

(b) Given a n-player game G, show how to construct a linear program (without any objective function), such that the set of feasible solutions of the linear program is precisely the set of CE of the game G. Use this to conclude that the set of CE of the game G must form a convex set (in fact a convex polytope), meaning that any convex combination (i.e., weighted average) of CEs of G is itself a CE of G.

(c) Consider a 2-player bimatrix game, given by:

$$\begin{bmatrix} (5,2) & (0,0) \\ (0,0) & (2,5) \end{bmatrix}$$

We can interpret this game as follows: two people want to meet each other, at one of two locations, `a` or `b`. Player 1 prefers to meet at `a`, giving him payoff 5, whereas if they meet at `b` he gets payoff 2. Player 2 prefers to meet at `b`, giving her payoff 5, otherwise if they meet at `a` she gets payoff 2. The two of them definitely do want to meet somewhere (they really like each other), because if they don't meet (i.e., if they go to different locations), they would both get payoff zero. The two can of course try to randomize their choice.

Compute a CE for this game, which is *not* a Nash equilibrium, and which gives the same expected payoff to both players, and such that the total welfare (i.e., sum of the expected utilities to both players) in the CE is as high as in any Nash equilibrium. Conclude that the set of CEs constitutes a superset of the set of Nash equilibria.

Notice, by the way that the set of Nash equilibria of the above game does not form a convex set. (In fact there are exactly 3 Nash equilibria for the above game, and thus clearly, not all convex combinations of them form a Nash equilibrium. Otherwise, there would be infinitely many Nash equilibria.)

(d) (Vague question) can you imagine any situations where a CE may make more "sense" as a solution concept than a Nash equilibrium? What about in the game above?