Sample Solution for AGTA Tutorial 8

1. Suppose you are running a VCG-based simultaneous multi-item auction, where three related items A, B, and C, are being auctioned simultaneously, and each bidder can bid on any possible subset of the items. Suppose there are two bidders, X and Y, and they provide you with their "claimed valuation" as their bids for every subset of the items, as part of the bidding process. Suppose that the valuation functions v_X and v_Y that you receive from the two bidders, X and Y, respectively, are as follows (the numbers denote millions of pounds):

	valuation							
bidder i	$v_i(\emptyset)$	$v_i(A)$	$v_i(B)$	$v_i(C)$	$v_i(\{A,B\})$	$v_i(\{A,C\})$	$v_i(\{B,C\})$	$v_i(\{A,B,C\})$
$i := X$	0	24	4	9	29	38	20	50
$i := Y$	0	15	18	11	30	34	32	47

- What is the outcome of this VCG auction? In other words, which of the two bidders will get which of the item(s), and what price will they each pay?

  Solution: Assume the bidders bid their true valuations (which we can, since in the VCG mechanism bidding the true valuations is a dominant strategy for all bidders).

  Then given these valuations, the VCG mechanism firstly picks an outcome that maximizes the sum total valuation of all the bidders, i.e., maximizes the total social welfare of the outcome. In this case, interestly, there are two completely different outcomes that maximize the social welfare. Namely, if player (bidder) X gets allocated items $\{A, C\}$ and player Y gets allocated $\{B\}$, then the sum total valuation of this outcome is:

  $$v_X(\{A,C\}) + v_Y(\{B\}) = 38 + 18 = 56.$$

  Likewise, if X gets $\{A\}$ and Y gets $\{B, C\}$ then the total valuation of this other outcome is:

  $$v_X(\{A\}) + v_Y(\{B,C\}) = 24 + 32 = 56.$$

  Furthermore, one can check (exhaustively) that these two outcomes provide the maximum total valuation of any outcome.

  Therefore, the VCG mechanism picks *one of these possible outcomes*, and then asks the bidders to pay the VCG payments associated with that outcome. Note that the VCG mechanism, as given, does not specify which of these outcomes is to be preferred. Either will do (and this does not change the incentive structure of the mechanism).

We note that these two outcomes are very different, and also yield very different payments. Specifically, assume the outcome that gives X the set $\{A, C\}$ and gives Y the set $\{B\}$.

We now calculate the payments of X and Y for this outcome: Note that, since Y is the only player other than X, the VCG payment for X in this outcome is $(\max_{O \in outcomes} v_Y(O)) - v_Y(\{B\})$. Now, clearly, the maximum valuation of any outcome O for Y is $v_Y(\{A, B, C\}) = 47$. Thus, the payment for X is $47 - 18 = 29$. Similarly, the payment for Y is $(\max_{O \in outcomes} V_X(O)) - V_X(\{A, C\}) = 50 - 38 = 12$.

On the other hand, if the VCG mechanism chooses the other social-welfare-maximizing outcome, namely X gets $\{A\}$ and Y gets $\{B, C\}$, then the payments are as follows: the payment of X is $(\max_{O \in outcomes} v_Y(O)) - v_Y(\{B, C\}) = 47 - 32 = 15$. the payment of Y is $(\max_{O \in outcomes} V_X(O)) - V_X(\{A\}) = 50 - 24 = 26$.

It is worth noticing the following fact about the prices paid by the bidders in the two different VCG outcomes in this auction.

Namely, in first VCG outcome, the sum total of the prices paid is

$$29 + 12 = 41$$

and in the second VCG outcome, the sum total of the prices paid is

$$15 + 26 = 41$$

In fact, this is not a co-incidence: in *every* VCG outcome the sum total of the payments by all of the agents must necessarily be the same.

Here is a proof (thanks to the participants in Tutorial 8 for asking about this, and thanks to Finlay Pearson for posting this proof on Piazza):

Recall that in the VCG mechanism, the set $V = \{1, \ldots, n\}$ of n agents declare their valuation functions, $v'_i : C \to \mathbb{R}_{\geq 0}$, $i \in V$, over the set C of possible outcomes. an outcome, $c^* \in C$, is called a VCG outcome, iff:

$$c^* \in \arg\max_{c \in C} \sum_{j \in V} v'_j(c)$$

In other words, a VCG outcome c^* maximizes the sum total declared valuation of all the bidders.

Recall also that, by definition, the price paid by player i for a VCG outcome c^* is, by definition, given by the following expression:

$$p_i(c^*) := (\max_{c' \in C} \sum_{j \in V \setminus \{i\}} v'_j(c')) - \sum_{j \in V \setminus \{i\}} v'_j(c^*)$$

Suppose that in a particular VCG setting, there is more than one VCG outcome. In particular, suppose c^A and c^B are two different VCG outcomes in the same setting.

Let

$$Z := \max_{c \in C} \sum_{j \in V} v'_j(c) = \sum_{j \in V} v'_j(c^A) = \sum_{j \in V} v'_j(c^B)$$

be the maximum sum total valuation (which occurs under both the outcomes c^A and c^B). Let

$$W_i := \max_{c' \in C} \sum_{j \in V \setminus \{i\}} v'_j(c')$$

denote the maximum sum total valuation for all players other than player i. Finally, let $W := \sum_{i \in V} W_i$.

We claim that the sum total of the payments by all the bidders in any two VCG outcomes, c^A and c^B, is always the same. Specifically, we claim it is always $W - (n-1)Z$. In other words, we claim:

$$\sum_{i \in V} p_i(c^A) = \sum_{i \in V} p_i(c^B) = W - (n-1)Z$$

To see this, we simply use the expression defining the prices, and do the sum:

$$\begin{aligned}
\sum_{i \in V} p_i(c^A) &= \sum_{i \in V} ((\max_{c' \in C} \sum_{j \in V \setminus \{i\}} v'_j(c')) - \sum_{j \in V \setminus \{i\}} v'_j(c^A)) \\
&= \sum_{i \in V} (\max_{c' \in C} \sum_{j \in V \setminus \{i\}} v'_j(c')) - \sum_{i \in V} \sum_{j \in V \setminus \{i\}} v'_j(c^A) \\
&= W - \sum_{i \in V} \sum_{j \in V \setminus \{i\}} v'_j(c^A) \\
&= W - ((\sum_{i \in V} \sum_{j \in V} v'_j(c^A)) - \sum_{i \in V} v'_i(c^A)) \\
&= W - (n \sum_{j \in V} v'_j(c^A) - \sum_{j \in V} v'_j(c^A)) \\
&= W - (nZ - Z) \\
&= W - (n-1)Z
\end{aligned}$$

Note that neither W nor Z depend on the specific VCG outcome c^A, and hence also the expression $W - (n-1)Z$ is independent of the specific VCG outcome. Hence, using the exact same derivation, we could have also derived that $\sum_{i \in V} p_i(c^B) = W - (n-1)Z$. In other words, in every VCG outcome, the sum total of all the prices paid by all the bidders (which corresponds to the *revenue* of the auctioneer if the VCG mechanism is being used in the setting of a multi-item auction) is the same, namely $W - (n-1)Z$.

- Do you expect the bidders to tell you the truth about their valuations?

Solutions: Yes. Despite the fact that there are two entirely different VCG outcomes in this multi-item auction, and despite the facts that the payments of the two bidders are completely different in the two outcomes, nevertheless, for both bidders it is a (weakly) dominant strategy to bid their true valuation functions, regardless of which total-value-optimal outcome is chosen by the VCG mechanism.

(This can be checked by following the proof given in class that the VCG mechanism is incentive compatible. That proof never uses an assumption that the optimal outcome is unique.)