1 Question 1

Consider a game graph $G = (V, E, v_0, F)$, where the (finite) set of vertices $V = V_1 \cup V_2$ is partitioned into set of vertices $V_1$ (belonging to player 1) and set of vertices $V_2$ (belonging to player 2), $E$ being the set of edges of $G$. We are also given a start vertex $v_0 \in V$, and a set $F \subset V$ of good (target) states. Denote $E(v)$ the set of all successor vertices for $v$, i.e $E(v) = \{v' \in V | (v, v') \in E\}$. We assume that $\forall v, E(v) \neq \emptyset$, i.e every state has a successor state (no deadlocks).

Let $\Pi$ denote the set of infinite paths in $G$. For $\pi \in \Pi$, $\pi = v_0v_1...$, let us denote the set of states that appear infinitely often in $\pi$ as $\text{inf}(\pi) = \{v \in V | \forall i \geq 0, \exists j \geq i, v_i = v\}$.

The play $\pi$ is a win for player 1 if $\text{inf}(\pi) \cap F \neq \emptyset$, and otherwise it is a win for player 2 (i.e a loss for player 1). Describe an efficient algorithm to find which player wins, and to extract a memoryless winning strategy, given such a game.

1.1 An Algorithm

For any set of nodes $S \subseteq V$, let $\text{Win}'_1(S)$ denote the set of nodes $v \in V$ such that, starting from $v$ player 1 has a winning strategy to force the game to reach a vertex in $S$ in one or more steps (so, we do not necessarily have $S \subseteq \text{Win}'_1(S)$).

It is easy to adapt the algorithm given in class for the win-lose reachability game on a graph (see the slides for Lecture 12, page 8), to compute the set $\text{Win}'_1(S)$.

Namely, in that algorithm, in step 1., instead of initializing $\text{Win}_1 := \text{Good}$, we simply instead initialize $\text{Win}_1$ as follows

$$\text{Win}_1 := \{ v \in V_1 | \exists (v, v') \in E \text{ such that } v' \in S \} \cup \{ v \in V_2 | \forall (v, v') \in E \text{ } v' \in S \}$$

(In the above $V_1$ denotes the vertices of the game graph belonging to player 1, and $V_2$ denotes the vertices belonging to player 2.)

The rest of that algorithm remains unchanged. This revised version of that algorithm will compute precisely the set $\text{Win}'_1(S)$. Note that algorithm also computes, a memoryless strategy for player 1, such that using that strategy, starting from every vertex $v \in \text{Win}'_1(S)$ the play will reach a vertex in $S$ (no matter what player 2 does).

Suppose $F$ is the set of “target” vertices that player 1 would like to visit infinitely often in the game. Let us now consider the following sequence of subsets of $F$.

Let $F_0 := F$, and for all integers $i \geq 0$, let

$$F_{i+1} := F \cap \text{Win}'_1(F_i)$$
In other words, $F_i$ denotes the set of target vertices $v \in F$ such that player 1 has a strategy to revisit target vertices in $F$ at least $i$ times, starting from $v$.

Note that we can computing the sets $F_{i+1}$ by just repeatedly using reachability game algorithm to compute the sets $Win'_1(F_i)$.

It is easy to show (by induction on $i$) that:

$$F = F_0 \supseteq F_1 \supseteq F_2 \supseteq F_3 \ldots$$

In other words, $F_i \supseteq F_{i+1}$, for all $i \in \mathbb{N}$.

Thus, for each $i \in \mathbb{N}$, either $F_i \subset F_{i+1}$, in which case $|F_{i+1}| \leq |F_i| - 1$, or else $F_i = F_{i+1}$. Now, notice that $F$ is a finite set. Thus if $|F| = k$ then, clearly there is some smallest $i \leq k$ such that $F_i = F_{i+1} = F_{i+2} = \ldots$.

Let $F^*$ be equal to $F_i$ for (the smallest) $i \leq k$ such that $F_i = F_{i+1}$.

But then, player 1 has a memoryless strategy, using which, starting at every $v \in F^*$, the play reaches a vertex in $F^*$ in one or more steps. Hence in fact (by reusing those same memoryless strategies starting from each node in $F^*$), starting from every vertex $v \in F^*$ player 1 has a strategy to infinitely often visit a node in $F^*$.

Finally, $F^* \cup Win'_1(F^*)$ is the set of all vertices (including vertices not in $F$) starting from which player 1 has a winning strategy to force visiting vertices in $F$ infinitely often.

The algorithm for solving a reachability game on a graph (i.e., computing a set $Win'_1(S)$), can be done in linear time $O(|E| + |V|)$ in the size of the game graph. Thus computing each $F_i$ at iteration $i$ of the algorithm requires $O(|E| + |V|)$ running time. Since there are at most $|F| = k$ iterations, the running time of the algorithm is $O(|F|(|E| + |V|))$. 

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