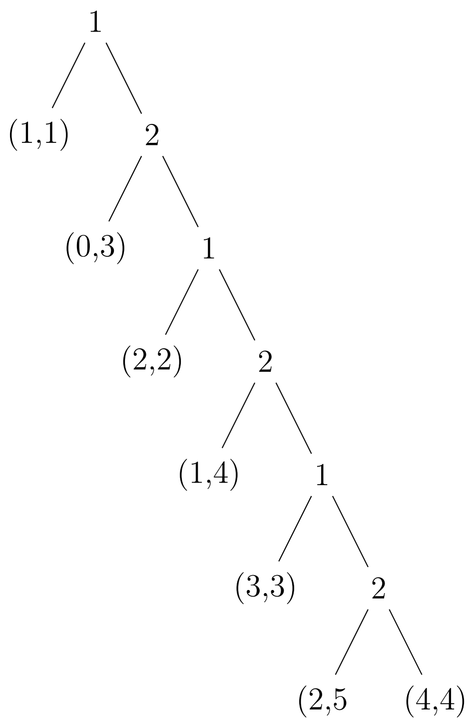
Tutorial 5: solution sketches

1. (a) Going left in the tree indicates stopping, and going right indicates giving.



   (b) Recall that a strategy for Player i function that tells Player i what to do at each node controlled by them. In this game we can write strategies just as tuples, so e.g. (G,G,S) is the strategy of giving twice and then stopping.

   One find the SPNE by backwards induction. At the last step, stopping is strictly dominating over giving for player 2. Knowing this, in the step before giving is stricltу dominated by stopping. And so forth until the very beginning. Thus the SPNE is given by ((S,S,S),(S,S,S)).

   (c) Working backwards in the above argument, we see that at each stage the choice strictly dominates the other option, so informally there is no "wiggle room". What this actually means is that the last game has a unique NE, and thus the second to last game has a unique SPNE and so forth. In short, the SPNE is unique.

   (d) Note that ((S,S,S),(S,S,S)) is a Nash Equilibrium for two reasons: P1 starts with S, so P2 can't improve (indeed, their choice doesn't matter) and P2 starts with S, so P1 can't improve. The remaining choices in the strategies don't affect whether this is a NE, only whether it is a SPNE. Thus any pair of strategies of the form ((S,-,-),(S,-,-)) is a pure NE for the game. We do not calculate the mixed NEs, but note that for example any pair of mixtures of such strategies is again an NE.

(e) Intuitively, if I could somehow commit say to (G,G,G) or even (G,G,S), the other players best response would give me (and them) a better payoff than just playing the SPNE. Likewise, if they could commit and I could trust them, it would be reasonable for me to play something like (G,G,S).

2. First of all, it is clear that Player 1 will always choose B whenever facing the choice at the leftmost node. Thus, we can and will from now on assume that player 1 will always play B in that leftmost subgame. Thus with 1/3 probability, the payoff to player 1 will be 3, and the payoff to player 2 will be 2. This is in fact the only proper subgame of the game, as a subgame must consist of a subtree with self-contained information sets, and say starting from player 2s information set doesn't form a subtree (it is a forest). Now let us consider the expected payoff overall, to both players. In effect, let us construct the normal form game corresponding to this extensive form game, after the action B at the leftmost node for player 1 has been fixed.

   It is not difficult to calculate the expected payoffs to both players under the remaining combinations of pure strategies (actions) for both players.

   Specifically, we get the following payoff table:

	a	b
BC	$((3+5+9)/3,(2+7+2)/3)$	$((3+5+5)/3,(2+7+2)/3)$
BD	$((3+10+6)/3,(2+3+6)/3)$	$((3+4+6)/3,(2+0+6)/3)$

   Or equivalently,

	a	b
BC	$(17/3,11/3)$	$(13/3,11/3)$
BD	$(19/3,11/3)$	$(13/3,8/3)$

   To see the above, note that, for example, if Player 1 plays B and C and player 2 plays "a" then the expected utility (payoff) for Player 1 is

$(3+5+9)/3 = 17/3$. We can likewise calculate all of the entries of the above table. (Note that in all these entries, it is always assumed that in the leftmost subtree player 1 plays B, because that is the unique optimal action in that subgame. So, without loss of generality, we can assume player 1 has two possible pure strategies: BC and BD, and of course it can also mix (randomize) between these two strategies.)

Now that we have the above normal form, we can easily calculate the Nash equilibria in this game, all of which will be "subgame perfect", because they already incorporate the fact that player 1 plays B in the leftmost subgame.

Note, in particular, that $((BD),(a))$ is a SPNE for the game, by inspection of the above payoff table: neither player can improve its payoff by switching strategies. Likewise $((BC),(b))$ is also an SPNE for the game, since both players can not *strictly* improve their payoff by unilaterally switching their strategy.

It is also not diffcult to check that there are no other, mixed NEs in this 2×2 normal form game. This is because as soon as player 1 puts positive probability on BD, it is preferrable for player 2 to switch its strategy to put probability 1 on pure strategy "a". Likewise, as soon as player 2 puts any positive probability on strategy "a", it is preferable for player 1 to put probability 1 on pure strategy BD.

The above two (pure) Nash Equilibria are both subgame perfect. So, there are exactly two SPNEs, both of which are pure.

Moreover, there are no other Nash Equilibria of any kind in the game. The reason is that, firstly, the only proper "subgame" of this game is the one in the leftmost subtree, rooted at the node controlled by player 1. But since there is a $1/3$ probability that the game will end up in that subgame, player 1 MUST play B with probability 1 in that subgame. Otherwise, if it puts positive probability on the action A, then it can always increase its own expected payoff (no matter what the other player does), by playing action B with probability 1 in that subgame. Hence, in all Nash equilibria (not just in all subgame perfect Nash equilibria), player 1 plays the action B with probability 1 in the leftmost subgame. Hence, there are no other NEs, other than the two pure NEs we have mentioned above.