AGTA Tutorial sheet 4 (solutions) Radu Ciobanu

1 Question 1

Given the finite 2 player bimatrix game from below, we will use iterated elimination of *strictly* dominated strategies.

$$\begin{array}{c c} & \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \begin{pmatrix} (5,2) & (22,4) & (4,9) & (7,6) \\ (16,4) & (18,5) & (1,10) & (10,2) \\ (15,12) & (16,9) & (18,10) & (11,3) \\ (9,15) & (23,9) & (11,5) & (5,13) \end{pmatrix} \end{array}$$

Consider Player 1 = Row player and Player 2 = Column player. We know that a pure strategy can be dominated by either a pure strategy or a *mixed* one. Observe that c_2 is strictly dominated by $(\frac{1}{2}c_1; \frac{1}{2}c_3)$ (easy to check). Therefore, we prune strategy c_2 and obtain a 4×3 residual game.

$$\begin{array}{c c} & \begin{array}{ccc} c_1 & c_3 & c_4 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \begin{pmatrix} (5,2) & (4,9) & (7,6) \\ (16,4) & (1,10) & (10,2) \\ (15,12) & (18,10) & (11,3) \\ (9,15) & (11,5) & (5,13) \end{pmatrix} \end{array}$$

Observe that r_1 and r_4 are strictly dominated by the pure strategy r_3. Also, from the new residual game, strategy c_4 is strictly dominated by c_3. The final residual game is:

$$\begin{array}{c c} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r_2 \\ r_3 \end{array} & \begin{pmatrix} (16,4) & (1,10) \\ (15,12) & (18,10) \end{pmatrix} \end{array}$$

Applying the same principles as in question 2 from tutorial 2, we obtain the unique Nash equilibrium $((\frac{1}{4}r_2; \frac{3}{4}r_3); (\frac{17}{18}c_1; \frac{1}{18}c_2))$. Therefore, the final answer: $((0; \frac{1}{4}; \frac{3}{4}; 0); (\frac{17}{18}; \frac{1}{18}; 0; 0))$

2 Question 2

2.1 a)

Suppose $x = (x_1, x_2, ...x_n)$ is a Nash equilibrium. Consider the product distribution $p(s_1, s_2, ...s_n) = \prod_{i=1}^{n} x_i(s_i)$. We will show that any Nash equilibrium is also a CCE.

$$U_i(x) = \sum_{(s_1,s_2,...s_n)\in S} u_i(s_1, s_2, ...s_n) * \prod_{i=1}^{n} x_i(s_i)$$

Therefore, since x is a N.E., from the claim in Lec.3, $\forall i, s_i' \in S_i$,

$$\sum_{(s_1,s_2,...s_n)\in S} u_i(s_1, s_2, ...s_n)*\prod_{i=1}^{n} x_i(s_i) \geq \sum_{(s_1,s_2,...s_n)\in S} u_i(s_1, s_2, ..s_{i-1}, s_i', ..s_n)*\prod_{i=1}^{n} x_i(s_i)$$

which is exactly the definition of a CCE where the probability distribution is $p(s_1, s_2, ...s_n) = \prod_{i=1}^{n} x_i(s_i)$.

From the Nash theorem, that says that in any finite $n - player$ game there must be at least one N.E, we conclude that it must also exist at least one CCE.

2.2 b)

From the definition in the tutorial sheet, the CCE could be written as the following LP problem (without objective):

$$\sum_{(s_1,s_2,...s_n)\in S} [u_i(s_1, s_2, ...s_n) - u_i(s_1, ...s_{i-1}, s_i', s_{i+1}...s_n)]*p(s_1, ...s_n) \geq 0, \forall i, s_i' \in S_i \tag{1}$$

$$p(s_1, ..., s_n) \geq 0, \forall (s_1, ..., s_n) \in S \tag{2}$$

$$\sum_{(s_1,s_2,...s_n)\in S} p(s_1, ..., s_n) = 1, \tag{3}$$

Now we show that any convex combination of CCEs of G is itself a CCE of G. In order to do this, we prove the following remark:

Lemma 2.1. *The set $\{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{Ax} \geq \boldsymbol{b}\}$ is convex, where $\boldsymbol{A}$ is an $m \times n$ matrix, $\boldsymbol{x}$ is an $n \times 1$ vector of variables and b is an $m \times 1$ vector of constants.*

Proof. From the definition of convexity, a set $D \subseteq \mathbb{R}^n$ is convex if for any 2 points u and $v \in D$, $\forall \lambda \in [0, 1]$, the point $\lambda u + (1 - \lambda)v \in D$.

Let $\mathbf{u}$ and $\mathbf{v}$ be members of the original set whose convexity we want to prove, i.e $\mathbf{u}$ and $\mathbf{v}$ are solutions to the linear system of inequalities $\mathbf{Ax} \geq \mathbf{b}$. In this case, $\mathbf{Au} \geq \mathbf{b}$ and $\mathbf{Av} \geq \mathbf{b}$. Hence, $\mathbf{A}(\lambda\mathbf{u} + (1-\lambda)\mathbf{v}) = \lambda\mathbf{Au} + (1-\lambda)\mathbf{Av} \geq \lambda\mathbf{b} + (1 - \lambda)\mathbf{b}$. Therefore, $\mathbf{A}(\lambda\mathbf{u} + (1 - \lambda)\mathbf{v}) \geq \mathbf{b}$. □

Using lemma 2.1, any weighted average of CCEs is itself a CCE, by using a suitable matrix $\mathbf{A}$ in (1), $\mathbf{x}$ is a vector of probabilities and $\mathbf{b}$ the null column.

2.3 c)

Given the game

$$\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} (5,2) & (0,0) \\ (0,0) & (2,5) \end{pmatrix} \end{array}$$

The N.E are:

- $[(\frac{5}{7}, \frac{2}{7}), (\frac{2}{7}, \frac{5}{7})]$ with expected payoff both for pl.1 and for player 2 : $\frac{10}{7}$.
- $[(0, 1), (0, 1)]$ with exp. payoff for player $1 = 2$ and exp. payoff for player $2 = 5$.
- $[(1, 0), (1, 0)]$ with exp. payoff for player $1 = 5$ and exp. payoff for player $2 = 2$.

Assign probabilities $p_{aa}, p_{ab}, p_{ba}, p_{bb}$, where $p_{aa} + p_{ab} + p_{ba} + p_{bb} = 1$.
Expected payoff for player 1 : $5p_{aa} + 0p_{ab} + 0p_{ba} + 2p_{bb} = 5p_{aa} + 2p_{bb}$
Likewise, expected payoff for player 2: $2p_{aa} + 5p_{bb}$.

Equalizing them , we obtain : $p_{aa} = p_{bb}$. Since both would like to meet, no player would prefer to go to different locations. In this case , CCE, $p_{ab} = p_{ba} = 0$. Therefore $p_{aa} = p_{bb} = \frac{1}{2}$. Hence $\mathbf{p} = (\frac{1}{2}, 0, 0, \frac{1}{2})$. Observe that the expected payoff for both players is 3.5, and the total expected utility is 7, being as high as in any Nash equilibrium.

Check easily that the set of NE's in this game is not convex, i.e a weighted average of N.E's is not a Nash equilibrium. Give a numerical example, by applying the definition of a convex set. Take $\lambda = \frac{1}{3}$ for instance.

2.4 d)

Open discussion. Relevant arguments might include:

- the CCE could give higher expected utilities than a Nash equilibrium; in the game above, for the CCE given as example, the total welfare is as high as any one given by a N.E.
- CCE is less computationally expensive
- Nash equilibrium is relevant if one assumes that each player knows which strategies the other players are using. In CCE, we do not make such assumption, players not knowing in general how others are playing. CCE does not require any explicit randomization on the part of the players. Each player always chooses a pure strategy with no attempt to randomize. The probabilistic nature of the strategies reflects the uncertainty of other players about his choice.
- NE $\subseteq$ CCE, i.e CCE is a superset of NE