Tutorial 3: solution sketches

1. Claim: $A = -A^T$ implies $x^T Ay = -y^T Ax$ for all vectors x, y of the right length.

   Proof. $x^T Ay = x^T(-A^T)y = -(x^T A^T y) = -(x^T A^T y)^T = -y^T Ax$, where the second to last step uses the fact that $B^T = B$ for all 1×1-matrices, and the last step uses the facts that $(B^T)^T = B$ and $(BC)^T = C^T B^T$. (One could of course prove the claim by e.g. direct calculation) $\square$

   In particular, the claim implies that $x^T Ax = -x^T Ax$, which gives $x^T Ax = 0$. This means that whenever both players play with the same mixed strategy x, they both have an expected payoff of zero. Thus in any strategy profile (x, y), if one of the players has a negative expected payoff, they can improve by copying the other players strategy. Thus no strategy profile giving non-zero expected payoffs can be a Nash equilibrium of the game.

2. Using the general recipe for LP duals (from page 6 of the slides for lecture 7 on LP duality), we get the following linear program for the dual LP:

   The variables of the dual LP are $y = (y_1, y_2, y_3, y_4)$. (Note that there is one variable in the dual corresponding to each inequality and equality constraint of the primal, but excluding the non-negativity constraints of the primal.)

   The

   Mimimize y_4
   Subject to:
   $-2y_1 - 9y_2 - 4y_3 + y_4 \geq 0$
   $-7y_1 + 0y_2 - 3y_3 + y_4 \geq 0$
   $y_1 + y_2 + y_3 = 1$
   $y_i \geq 0$ for $i = 1, 2, 3$

   Which is easily seen to be equivalent to the LP for computing the value and the maxminimizer strategy for player 2. (The variable y_4 represents the value, which player 2 is trying to minimize, and the mixed strategy of player 2 is represented by (y_1, y_2, y_3).)

   There are a number of ways we can solve this LP. We obtain that its optimal value (the value of the game) is $y_4 = \frac{11}{3}$, and where the unique maxminimizer profile for player 2 is (y_1, y_2, y_3), where: $y_1 = \frac{1}{6}$, $y_2 = 0$, $y_3 = 5/6$.

3. Let's proceed as in the hint. b_i is the amount of units of vitamin i you need daily. For simplicity let's say the b_i is measured in "grams-of-vitimin-i" (it doesn't matter

if it is grams or some other unit). Likewise, for simplicity, let us assume that the foods are also measured in units of grams.

The entries a_{ij} in the matrix A measure the number of units of vitamin i per unit of food j, or in other words "gram-of-vitamin i/gram-of-food-j". The number c_j is the cost-per-unit of food j, so it is measured in £/gram-of-food-j.

Now, the dual LP is

Maximize $b^T y$
Subject to:
$A^T y \leq c$
$y_i \geq 0$

Since c_i is measured in £/gram-of-food-j, the left hand side of each constraint in $A^T y \leq c$ must also be in the same units. So, in particular, the j'th constraint $(A^T y)_j = \sum_{i=1}^{m} a_{i,j} y_i \leq c_j$ must have each $a_{i,j} y_i$ be in units of £/gram-of-food-j. Since $a_{i,j}$ is in units of "gram-of-vitamin i/gram-of-food-j", we must have y_i in units of £/gram-of-vitimin-i. Furthermore, each term $b_i y_i$ in the objective function $b^T y$ is thus measured in grams-of-vitimin-i $*$ £/gram-of-vitimin-i $=$ £. So, the objective function is somehow trying to maximize the amount of money (earned), and the variables y_i are setting prices per-unit (i.e, per gram) of each vitimin i.

But what are the constraints $(A^T y)_j \leq c_j$, saying?

They are saying that the amount of money charged per-unit of each vitamin should be such that, if the number of units of each vitamin i contained in each until of food j is as specified by the entry $a_{i,j}$ in the matrix A, then the cost-per-unit of food j should not be less than the total costs per unit of all the vitamins that are in one unit of food j.

So, we can in effect view this dual problem as the "vitamin seller's problem". To be more precise, suppose the <u>only</u> thing that costs money in the production of food j is the cost for the amount of each vitamin that is contained in each unit of food j. (In fact, for simplicity, we can simply assume that each unit of food j consists of nothing other than the units of all the different vitamins that are contained in it.)

Suppose that the price per unit, c_j, for each food has been pre-determined somehow.

The vitamin seller needs to set a price per-unit for each vitamin, in such a way as to do both of the following:

- make sure that the total cost-per-unit c_j of food j is not strictly less that the total cost that would be required to buy all the units of vitamins that are supposedly contained in each unit of food j.

(These are the constraints $A^T y \leq c$.)

- maximize $b^T y$, which measures the total income (to the vitamin seller) if the buyer (who is buying the foods), buys *just enough* of each food to get a total of exactly the minimum required units, b_i, of each of the n vitamins.

So, this "Vitamin seller's" price setting problem is the problem that the dual LP is trying to solve.

It is indeed interesting that (by the strong duality theorem) the maximum income the vitamin seller can achieve for itself by setting these prices while satisfying these constraints (namely, making sure each food can afford the vitamins in it, based on its price-per-unit c_j), is the same as the minimum cost incurred by the buyer if it tries to minimize its own cost, subject to getting at least the minimum required b_i units of each vitamin i.

Note also that by the "Complementary Slackness" theorem it follows that, in a pair of optimal solutions for the buyer and the seller, either the price y_i of a vitamin i is set at 0, or else the amount of food the buyer buys contains exactly the minimum required b_i units of vitamin i, i.e., $(Ax)_i = b_i$. Likewise, either the amount x_i of food j purchased by the buyer is equal to 0, or else the total cost of the vitamins $(A^T y)_j$ contained in each unit of food j (at the prevailing prices y of the vitamins) is exactly equal to c_j (which is the maximum allowable cost per unit of food j).