Tutorial 3: sample solutions

1. We first establish the following:

   Claim: $A = -A^T$ implies $x^T Ay = -y^T Ax$ for all vectors x, y of the right length.

   Proof. $x^T Ay = x^T(-A^T)y = -(x^T A^T y) = -(x^T A^T y)^T = -y^T Ax$, where the second to last step uses the fact that $B^T = B$ for all 1×1-matrices, and the last step uses the facts that $(B^T)^T = B$ and $(BC)^T = C^T B^T$. (One could of course prove the claim by e.g. direct calculation) $\square$

   In particular, the claim implies that $x^T Ax = -x^T Ax$, which gives $x^T Ax = 0$. This means that whenever both players play with the same mixed strategy x, they both have an expected payoff of zero. Thus in any strategy profile (x, y), if one of the players has a negative expected payoff, they can improve by copying the other players strategy. Thus no strategy profile giving non-zero expected payoffs can be a Nash equilibrium of the game.

2. Using the recipe from page 12 of the slides for lecture 4, we get the linear program

   Maximize v
   Subject to:
   $(x^T A)_j \geq v$
   $\sum_i x_i = 1$
   $x_i \geq 0$

   Maximize v
   Subject to:
   $2x_1 + 7x_2 \geq v$
   $9x_1 + 0x_2 \geq v$
   $4x_1 + 3x_2 \geq v$
   $x_1 + x_2 = 1$
   $x_1 \geq 0, x_2 \geq 0$

   Writing this out explicitly, we get the the linear program

   which is equivalent to the linear program:

   Maximize v
   Subject to:
   $v - 2x_1 - 7x_2 \leq 0$
   $v - 9x_1 - 0x_2 \leq 0$
   $v - 4x_1 - 3x_2 \leq 0$

$x_1 + x_2 = 1$
$x_1 \geq 0, x_2 \geq 0$

Note that, because we have $x_1 + x_2 = 1$, we can express x_2 as $x_2 = (1 - x_1)$. We can then replace all occurences of x_2 in the constraints, to obtain the following "equivalent" new LP:

Maximize v
Subject to:
$v - 2x_1 - 7(1 - x_1) \leq 0$
$v - 9x_1 - 0(1 - x_1) \leq 0$
$v - 4x_1 - 3(1 - x_1) \leq 0$
$x_1 + (1 - x_1) = 1$
$x_1 \geq 0, (1 - x_1) \geq 0$

which in turn is equivalent to:

Maximize v
Subject to:
$v + 5x_1 - 7 \leq 0$
$v - 9x_1 \leq 0$
$v - x_1 - 3 \leq 0$
$0 \leq x_1 \ , x_1 \leq 1$

We can solve this LP in a number of ways. Let us use Fourier-Motzkin elimination. In order to eliminate the variable x_1, we have to rewrite each inequality so that x_1 occurs on one side of the inequality. We get:

(1) **Maximize** v

Subject to:

$$\begin{aligned} x_1 &\leq \frac{7}{5} - \frac{1}{5}v \\ x_1 &\leq 1 \\ \frac{1}{9}v &\leq x_1 \\ v - 3 &\leq x_1 \\ 0 &\leq x_1 \end{aligned}$$

To elminate the variable x_1, we combine each of the two lower bound inequalities on x_1 with each of the three upper bound inequalities on x_1, to obtain the following six inequalities:

Maximize v
Subject to:
$\frac{1}{9}v \leq \frac{7}{5} - \frac{1}{5}v$
$v - 3 \leq \frac{7}{5} - \frac{1}{5}v$
$0 \leq \frac{7}{5} - \frac{1}{5}v$
$\frac{1}{9}v \leq 1$
$v - 3 \leq 1$
$0 \leq 1,$

By simplifying the inequalities, this can be equivalently expressed as:

Maximize v
Subject to:
$v \leq \frac{9}{2}$
$v \leq \frac{22}{6} = \frac{11}{3}$
$v \leq 7$
$v \leq 9$
$v \leq 4$
$0 \leq 1,$

Of all the above inequalities, the one that privides the smallest upper bound on v is the inequality $v \leq \frac{11}{3}$.

Hence, the maximum value we can obtain for v that satisfies all these inequalities is $v = \frac{11}{3}$.

We next use this value for $v = \frac{11}{3}$, and plug it back into the inequalities in (**??**), to recover the value of x_1. We get:

$x_1 \leq \frac{7}{5} - \frac{1}{5}\frac{11}{3}$
$x_1 \leq 1$
$\frac{1}{9}\frac{11}{3} \leq x_1$
$\frac{11}{3} - 3 \leq x_1$
$0 \leq x_1$,

These can be re-written as:

$x_1 \leq \frac{21}{15} - \frac{11}{15} = \frac{10}{15} = \frac{2}{3}$
$x_1 \leq 1$
$\frac{11}{27} \leq x_1$
$\frac{2}{3} = \frac{11}{3} - \frac{9}{3} \leq x_1$
$0 \leq x_1$,

Note that combining the first and fourth inequalities implies we must have $x_1 = 2/3$. Combining this with $x_2 = (1 - x_1)$ we get that we must have $x_2 = 1/3$.

Hence, the minimax value of this two player zero sum game is $\frac{11}{3}$, and furthermore the unique minmaximizer strategy for player 1 is $(2/3, 1/3)$.

(Can you think of an alternative way to establish the same thing, but which avoids using LP and Fourier-Motzkin elimination, and instead uses the "useful corollary to Nash's theorem"?)