Tutorial 3: sample solutions

1. We first establish the following:

Claim: $A = -A^T$ implies $x^T Ay = -y^T Ax$ for all vectors $x, y$ of the right length.

Proof. $x^T Ay = x^T(-A^T)y = -(x^T A^T y) = -(x^T A^T y)^T = -y^T Ax$, where the second to last step uses the fact that $B^T = B$ for all $1 \times 1$-matrices, and the last step uses the facts that $(B^T)^T = B$ and $(BC)^T = C^T B^T$. (One could of course prove the claim by e.g. direct calculation) \[Q.E.D.\]

In particular, the claim implies that $x^T Ax = -x^T Ax$, which gives $x^T Ax = 0$. This means that whenever both players play with the same mixed strategy $x$, they both have an expected payoff of zero. Thus in any strategy profile $(x, y)$, if one of the players has a negative expected payoff, they can improve by copying the other players strategy. Thus no strategy profile giving non-zero expected payoffs can be a Nash equilibrium of the game.

2. Using the recipe from page 12 of the slides for lecture 4, we get the linear program

Maximize $v$
Subject to:
\[
(x^T A)_j \geq v \\
\sum_i x_i = 1 \\
x_i \geq 0
\]

Maximize $v$
Subject to:
\[
2x_1 + 7x_2 \geq v \\
9x_1 + 0x_2 \geq v \\
4x_1 + 3x_2 \geq v \\
x_1 + x_2 = 1 \\
x_1 \geq 0, x_2 \geq 0
\]

Writing this out explicitly, we get the the linear program

which is equivalent to the linear program:

Maximize $v$
Subject to:
\[
v - 2x_1 - 7x_2 \leq 0 \\
v - 9x_1 - 0x_2 \leq 0 \\
v - 4x_1 - 3x_2 \leq 0
\]
\[ x_1 + x_2 = 1 \]
\[ x_1 \geq 0, x_2 \geq 0 \]

Note that, because we have \( x_1 + x_2 = 1 \), we can express \( x_2 \) as \( x_2 = (1 - x_1) \). We can then replace all occurrences of \( x_2 \) in the constraints, to obtain the following “equivalent” new LP:

**Maximize** \( v \)

**Subject to:**
\[ v - 2x_1 - 7(1 - x_1) \leq 0 \]
\[ v - 9x_1 - 0(1 - x_1) \leq 0 \]
\[ v - 4x_1 - 3(1 - x_1) \leq 0 \]
\[ x_1 + (1 - x_1) = 1 \]
\[ x_1 \geq 0, (1 - x_1) \geq 0 \]

which in turn is equivalent to:

**Maximize** \( v \)

**Subject to:**
\[ v + 5x_1 - 7 \leq 0 \]
\[ v - 9x_1 \leq 0 \]
\[ v - x_1 - 3 \leq 0 \]
\[ 0 \leq x_1, x_1 \leq 1 \]

We can solve this LP in a number of ways. Let us use Fourier-Motzkin elimination. In order to eliminate the variable \( x_1 \), we have to rewrite each inequality so that \( x_1 \) occurs on one side of the inequality. We get:

(1) \[
\text{Maximize } v \\
\text{Subject to: } \\
x_1 \leq \frac{7}{5} - \frac{1}{5}v \\
x_1 \leq 1 \\
\frac{1}{9}v \leq x_1 \\
v - 3 \leq x_1 \\
0 \leq x_1
\]

To eliminate the variable \( x_1 \), we combine each of the two lower bound inequalities on \( x_1 \) with each of the three upper bound inequalities on \( x_1 \), to obtain the following six inequalities:
Maximize \( v \)
Subject to:
\[
\begin{align*}
\frac{1}{9}v & \leq \frac{7}{5} - \frac{1}{5}v \\
v - 3 & \leq \frac{7}{5} - \frac{1}{5}v \\
0 & \leq \frac{7}{5} - \frac{1}{5}v \\
\frac{1}{9}v & \leq 1 \\
v - 3 & \leq 1 \\
0 & \leq 1,
\end{align*}
\]

By simplifying the inequalities, this can be equivalently expressed as:

Maximize \( v \)
Subject to:
\[
\begin{align*}
v & \leq \frac{9}{2} \\
v & \leq \frac{22}{5} = \frac{11}{3} \\
v & \leq 7 \\
v & \leq 9 \\
v & \leq 4 \\
0 & \leq 1,
\end{align*}
\]

Of all the above inequalities, the one that provides the smallest upper bound on \( v \) is the inequality \( v \leq \frac{11}{3} \).

Hence, the maximum value we can obtain for \( v \) that satisfies all these inequalities is \( v = \frac{11}{3} \).

We next use this value for \( v = \frac{11}{3} \), and plug it back into the inequalities in (??), to recover the value of \( x_1 \). We get:
\[
\begin{align*}
x_1 & \leq \frac{7}{5} - \frac{11}{5} \cdot \frac{3}{9} \\
x_1 & \leq 1 \\
\frac{11}{3} & \leq x_1 \\
\frac{11}{3} - 3 & \leq x_1 \\
0 & \leq x_1,
\end{align*}
\]

These can be re-written as:
\[
\begin{align*}
x_1 & \leq \frac{21}{15} - \frac{11}{15} = \frac{10}{15} = \frac{2}{3} \\
x_1 & \leq 1 \\
\frac{11}{3} & \leq x_1 \\
x_1 & = \frac{11}{3} - 0 \leq x_1 \\
0 & \leq x_1,
\end{align*}
\]

Note that combining the first and fourth inequalities implies we must have \( x_1 = 2/3 \).
Combining this with \( x_2 = (1 - x_1) \) we get that we must have \( x_2 = 1/3 \).
Hence, the minimax value of this two player zero sum game is $\frac{11}{3}$, and furthermore the unique minimaximizer strategy for player 1 is $(2/3, 1/3)$.

(Can you think of an alternative way to establish the same thing, but which avoids using LP and Fourier-Motzkin elimination, and instead uses the “useful corollary to Nash’s theorem”? )