

Tutorial 3: solution sketches

1. Claim: $A = -A^T$ implies $x^T A y = -y^T A x$ for all vectors x, y of the right length.

Proof. $x^T A y = x^T (-A^T) y = -(x^T A^T y) = -(x^T A^T y)^T = -y^T A x$, where the second to last step uses the fact that $B^T = B$ for all 1×1 -matrices, and the last step uses the facts that $(B^T)^T = B$ and $(BC)^T = C^T B^T$. (One could of course prove the claim by e.g. direct calculation) \square

In particular, the claim implies that $x^T A x = -x^T A x$, which gives $x^T A x = 0$. This means that whenever both players play with the same mixed strategy x , they both have an expected payoff of zero. Thus in any strategy profile (x, y) , if one of the players has a negative expected payoff, they can improve by copying the other players strategy. Thus no strategy profile giving non-zero expected payoffs can be a Nash equilibrium of the game.

2. Using the recipe from page 12 of the slides for lecture 4, we get the linear program

Maximize v

Subject to:

$$(x^T A)_j \geq v$$

$$\sum_i x_i = 1$$

$$x_i \geq 0$$

Writing this out explicitly, we get the the linear program **Maximize** v

Subject to:

$$2x_1 + 7x_2 \geq v$$

$$9x_1 + 0x_2 \geq v$$

$$4x_1 + 3x_2 \geq v$$

$$x_1 + x_2 = 1$$

$$x_1 \geq 0, x_2 \geq 0$$

which is equivalent to the linear program that was given. To compute the dual using the general recipe, we express it in the form

Maximize $c^T x$

Subject to:

$$(Bx)_i \leq b_i$$

$$(Bx)_j = b_j$$

$$x_i \geq 0$$

Using the given linear program we can get to the desired form by setting $b^T = (0, 0, 0, 1)$ $c^T = (0, 0, 1)$, $x^T = (x_1, x_2, v)$ and

$$B = \begin{bmatrix} -2 & -7 & 1 \\ -9 & 0 & 1 \\ -4 & -3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

To be clear with the indices, our linear program is then

Maximize $c^T x$

Subject to:

$$(Bx)_i \leq b_i \text{ for } i = 1, 2, 3$$

$$(Bx)_4 = b_4$$

$$x_i \geq 0 \text{ for } i = 1, 2$$

Now, using the general recipe the dual is

Minimize $b^T y$

Subject to:

$$(B^T y)_i \geq c_i \text{ for } i = 1, 2$$

$$(B^T y)_3 = c_3$$

$$y_i \geq 0 \text{ for } i = 1, 2, 3,$$

which, when setting $y = (y_1, y_2, y_3, v)$ translates to

Maximize v

Subject to:

$$-2y_1 - 9y_2 - 4y_3 + v \geq 0$$

$$-7y_1 + 0y_2 - 3y_3 + v \geq 0$$

$$y_1 + y_2 + y_3 = 1$$

$$y_i \geq 0 \text{ for } i = 1, 2, 3$$

Which is easily seen to be equivalent to the LP for computing the value and the maxminimizer strategy for player 2.

- Let's proceed as in the hint. b_i is the amount of units of vitamin i you need daily. It is reasonable to use units of weight for it, so let's say the b_i is measured in kilograms kg . Numbers in the matrix A are portions, so they are unitless. The numbers c_j are costs of different foods per unit, so let's use $\text{£}/kg$ as the unit.

Now, the dual LP is

Maximize $b^T y$

Subject to:

$$A^T y \leq c$$

$$y_i \geq 0$$

As c_i is measured in \mathcal{L}/kg , the first constraint does not make sense unless it is in the same units, so the variables must be measured in \mathcal{L}/kg as well (as the elements of the matrix were unitless) – so they can be viewed as some sort of prices per unit. Furthermore, the objective function is thus measured in $kg \cdot \mathcal{L}/kg = \mathcal{L}$. Assuming that a reasonable interpretation exists, the objective function must be someone trying to maximize their income (as no one would maximize their losses). Given all this, one might just get lucky and guess the correct interpretation: a vitamin merchant is trying to maximize their income. It remains to check that this interpretation does make sense. In this interpretation, the variable y_i is the price per unit of vitamin i , and the merchant is able to set these freely. Now the objective function $b^T y$ tells the cost per a buyer of getting all their daily vitamins they need from the merchant, so it is indeed reasonable for the merchant to try to maximise that function. The constraints make sense too: the vitamins must have non-negative prices, and a constraint of the form $(A^T y)_i \leq c_i$ just says that, for the same amount of vitamins, it must be cheaper (or at most as expensive) to buy them from the merchant than to consume food i . This is reasonable too, for otherwise consumers would just get their vitamins from food instead of buying it from the vitamin merchant. Now we've checked all aspects of the dual LP make sense from the point of view of the interpretation we guessed, so our interpretation seems to be a plausible answer.