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Tutorial 1: solution sketches

1. Let's first consider the tie-breaking rule in which all players who are closest to half the average split the payoff. We claim that the unique NE of the game is when all players guess 1. This situation is clearly a NE, for any player unilaterally changing their guess will get payoff 0 instead of $1/n$. To show uniqueness, we show that no other situation is a NE (pure or mixed). Let $x_1, \ldots x_n$ be a strategy profile (we allow mixed strategies), and let k be the largest number in the support of any of the x_i, and let x_i be a player who has the guess k in the support, i.e. $x_i(k) > 0$. As k is the largest number anyone might be guessing, it gives a nonzero payoff only when all the other players are guessing k as well, in which case the payoff. If $k > 1$, guessing say $k - 1$ instead would give a payoff of 1 in this case, and in all other cases would do at least as well as guessing k. Thus, if there is a chance of everyone guessing k, Player i would strictly increase their expected payoff by switching to x'_i, defined by $x'_i(k) = 0, x'_i(k-1) = x_i(k) + x_i(k-1)$ and $x'_i(j) = x_i(j)$ otherwise. If there is no chance of everyone guessing k, it is easy to see that Player 2 can strictly increase their expected payoff by doing something similar. This implies that no NE consists of a strategy profile in which someone has a number $k > 1$ in the support of their strategy.

   In the second tie-breaking rule, all players who are closest to half the average get the full payoff of 1. Intuitively, this means that a player wants to win, but doesn't care about how many other players are winning simultaneously with them, unlike in the previous version, where players prefer to win alone if they can. this means that we get more NEs. For any $k \in \{1, \ldots 1000\}$, all players outputting k is a pure NE, as everyone is getting the maximum payoff of 1 and thus cannot improve. We claim that there are no other NEs. First we consider the possibility of other pure NEs. Note that in no NE is any player getting expected payoff of zero. Thus in a pure NE everyone must be getting the expected payoff of 1, and it easy easy to check that there are no other ways of arranging that.

   As far as mixed ones go, the idea is similar as in the previous case. Assume that $x_1, \ldots x_n$ is a NE that is not pure, and let k be the largest number in the support of any of the x_i, and let i be one of the players with $x_i(k) > 0$. The idea is to show that $k > 1$ leads to contradiction, so there are no mixed NEs. Now, if any of the other players are not playing pure strategy k, there is a chance that player i gets nothing. Let $l < k$ be a guess that would have given a nonzero payoff to Player i in one of such cases. Now Player i can strictly improve their expected utility by "moving the weight from k to l", i.e. using the strategy x'_i defined by $x'_i(k) = 0, x'_i(l) = x_i(k) + x_i(l)$ and $x'_i(j) = x_i(j)$ otherwise. The remaining case has all the other players playing the pure strategy k, while Player i is not playing a pure strategy. Repeating the previous reasoning from the point of view of any other player then shows that $x_1, \dots x_n$ is not a NE, concluding the proof.

2. (a) The expected payoff for Player 1 is given by

$$\begin{aligned}
U_1(x) &= \sum_{i,j} x_1(i)x_2(j)u_1(i,j) = \sum_i \sum_j x_1(i)x_2(j)u_1(i,j) \\
&= \frac{1}{4}(\frac{2}{3}u_1(1,1) + \frac{1}{3}u_1(1,2)) + \frac{1}{2}(\frac{2}{3}u_1(2,1) + \frac{1}{3}u_1(2,2)) + \frac{1}{4}(\frac{2}{3}u_1(3,1) + \frac{1}{3}u_1(3,2)) \\
&= \frac{1}{4}(\frac{2}{3}\cdot 7 + \frac{1}{3}\cdot 6) + \frac{1}{2}(\frac{2}{3}\cdot 4 + \frac{1}{3}\cdot 5) + \frac{1}{4}(\frac{2}{3}\cdot 6 + \frac{1}{3}\cdot 3) \\
&= 61/12
\end{aligned}$$

(b) We use iterated elimination of strictly dominated strategies. First of all, strategy 4 of Player 2 strictly dominates strategies 1 and 2, resulting in the bimatrix

$$\begin{bmatrix} (5,5) & (4,7) \\ (8,6) & (5,8) \\ (2,4) & (5,9) \end{bmatrix}$$

In the remaining bimatrix, strategy 1 of Player 1 is strictly dominated by strategy 2, so we end up with the bimatrix

$$\begin{bmatrix} (8,6) & (5,8) \\ (2,4) & (5,9) \end{bmatrix}$$

Now the first strategy (third in the original game) of Player 2 is strictly dominated by strategy 2 (strategy 4 in the original game), so we get the bimatrix

$$\begin{bmatrix} (5,8) \\ (5,9) \end{bmatrix}$$

Player 1 will get expected payoff 5 no matter what they do in the remaining game, whereas Player 2 has no choices remaining. Thus in this reduced game, the set of Nash equilibria is $\{(x_1, x_2) \mid x_i \text{ a mixed strategy for Player } i \text{ and } (x_1, x_2) \text{ is a NE }\} = \{((p, 1-p), 1) \mid p \in [0,1]\}$. This means that in the original game the Nash Equilibria are given by pairs (x_1, x_2) where x_1 is of the form $(0, p, 1-p)$ and $x_2 = (0,0,0,1)$. In particular, we have two pure NEs: in one Player 1 plays strategy 2 hile Player 2 plays strategy 4 and in the other one Player plays strategy 3 while Player 2 plays strategy 4.