Algorithmic Game Theory and Applications

Lecture 9: Computing solutions for General Strategic Games: Part II: Nash Equilibria

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from last time: Computing Nash Equilibria: a first clue

Recall "Useful corollary for NEs", from Lecture 3:

> If x^* is an NE, then if $x_i^*(j) > 0$ then $U_i(x_{-i}^*; \pi_{i,j}) = U_i(x^*)$.

Using this, and adding a condition, we can fully characterize Nash Equilibria:

Proposition 1 In an n-player game, a profile x^* is a Nash Equilibrium if and only if there exist $w_1, \ldots, w_n \in \mathbb{R}$, such that the following hold:

1. For all players i, and every $\pi_{i,j} \in \mathit{support}(x_i^*)$, $U_i(x_{-i}^*; \pi_{i,j}) = w_i$, and
2. For all players i, and every $\pi_{i,j} \notin \mathit{support}(x_i^*)$, $U_i(x_{-i}^*; \pi_{i,j}) \leq w_i$.

Note: Any such w_i's necessarily satisfy $w_i = U_i(x^*)$.

Proof Follows easily from what we already know, particularly 1st claim in the proof of Nash's theorem.■

using our first clue

- Suppose we somehow know support sets, *support*$_1 \subseteq S_1, \ldots,$ *support*$_n \subseteq S_n$, for some Nash Equilibrium $x^* = (x_1^*, \ldots, x_n^*)$.
- Then, using Proposition 1, to find a NE we only need to solve the following system of constraints:

  1. For all players i, and every $\pi_{i,j} \in$ *support*$_i$, $U_i(x_{-i}; \pi_{i,j}) = w_i$,
  2. For all players i, and every $\pi_{i,j} \notin$ *support*$_i$, $U_i(x_{-i}; \pi_{i,j}) \leq w_i$.
  3. for $i = 1, \ldots, n$, $x_i(1) + \ldots + x_i(m_i) = 1$.
  4. for $i = 1, \ldots, n$, & for $j \in$ *support*$_i$, $x_i(j) \geq 0$.
  5. for $i = 1, \ldots, n$, & for $j \notin$ *support*$_i$, $x_i(j) = 0$.

- This system has $\sum_{i=1}^{n} m_i + n$ variables, $x_1(1), \ldots, x_1(m_1), \ldots, x_n(1), \ldots, x_n(m_n), w_1, \ldots, w_n$.
- Unfortunately, for $n > 2$ players, this is a non-linear system of constraints.
  Let's come back to the case $n > 2$ players later.
- Consider the 2-player case: the system is an LP!! But,
  Question: How do we find *support*$_1$ & *support*$_2$?
  Answer: Just guess!!

First algorithm to find NE's in 2-player games

Input: A 2-player strategic game Γ, given by rational values $u_1(s, s')$ & $u_2(s, s')$, for all $s \in S_1$ & $s' \in S_2$. (I.e., the input is $(2 \cdot m_1 \cdot m_2)$ rational numbers.)

Algorithm:

- For all possible *support*$_1 \subseteq S_1$ & *support*$_2 \subseteq S_2$:
  - Check if the corresponding LP has a feasible solution $x^*, w_1, \ldots, w_n$. (using, e.g., Simplex).
  - If so, STOP: the feasible solution x^* is a Nash Equilibrium (and $w_i = U_i(x^*)$).

Question: How many possible subsets *support*$_1$ and *support*$_2$ are there to try?

Answer: $2^{(m_1+m_2)}$

So, unfortunately, the algorithm requires worst-case exponential time.

But, at least we have our first algorithm.

remarks on algorithm 1

- The algorithm immediately yields:
  Proposition Every finite 2-player game has a rational NE. (Furthermore, the rational numbers are not "too big", i.e., are polynomial sized.)

- The algorithm can easily be adapted to find not just any NE, but a "good" one. For example:

  Finding a NE that maximizes "(util.) social welfare":

  - For each support sets, simply solve the LP constraints while maximizing the objective

    $$f(x,w) = w_1 + w_2 + \ldots + w_n$$

  - Keep track of best NE encountered, & output optimal NE after checking all support sets.

- The same algorithm works for any notion of "good" NE that can be expressed via a linear objective and (additional) linear contraints: (e.g.: maximize Jane's payoff, minimize John's, etc.)

- Note: This algorithm shows that finding a NE for 2-player games is in "**NP**".

Towards another algorithm for 2-players

Let A be the $(m_1 \times m_2)$ payoff matrix for player 1,
B be the $(m_2 \times m_1)$ matrix for player 2,
$\mathbf{w_1}$ be the m_1-vector, all entries $= w_1$,
$\mathbf{w_2}$ be the m_2-vector, all entries $= w_2$.

Note: We can safely assume $A > 0$ and $B > 0$: by adding a large enough constant, d, to every entry we "shift" each matrix > 0. Nothing essential about the game changes: payoffs just increase by d.

We can get another, related, characterization of NE's by using "slack variables" as follows:

Lemma $x^* = (x_1^*, x_2^*)$ is a NE if and only if:

1. There exists a m_1-vector $y \geq 0$, and $w_1 \in \mathbb{R}$, such that $$Ax_2^* + y = \mathbf{w_1}$$ & for all $j = 1, \ldots, m_1$, $x_1^*(j) = 0$ or $(y)_j = 0$.
2. There exists a m_2-vector $z \geq 0$, and $w_2 \in \mathbb{R}$, such that $$Bx_1^* + z = \mathbf{w_2}$$ & for all $j = 1, \ldots, m_2$, $x_2^*(j) = 0$ or $(z)_j = 0$.

Proof Again follows by the Useful Corollary to Nash: in a NE x^* whenever, e.g., $x_1^*(j) > 0$, $U(x_{-1}^*; \pi_{1,j}) = U(x^*)$. Let $(y)_j = U(x^*) - U(x_{-1}^*; \pi_{1,j})$. ▪

rephrasing the problem

The Lemma gives us some "constraints" that characterize NE's:

1. $Ax_2 + y = \mathbf{w_1}$ and $Bx_1 + z = \mathbf{w_2}$
2. $x_1, x_2, y, z \geq 0$.
3. x_1 and x_2 must be probability distributions, i.e., $\sum_{j=1}^{m_1} x_1(j) = 1$ and $\sum_{j=1}^{m_2} x_2(j) = 1$.
4. Additionally, x_1 and y, as well as x_2 and z, need to be "**complementary**":
   for $j = 1, \ldots, m_1$, either $x_1(j) = 0$ or $(y)_j = 0$,
   for $j = 1, \ldots, m_2$, either $x_2(j) = 0$ or $(z)_j = 0$.
   Since everything is ≥ 0, we can write this as

$$y^T x_1 = 0 \quad \text{and} \quad z^T x_2 = 0$$

continuing the reformulation

Note that, because $A > 0$ and $B > 0$, we know that $w_1 > 0$ and $w_2 > 0$ in any solution.

Using this, we can "eliminate" w_1 and w_2 from the constraints as follows: Let $x_2' = (1/w_1)x_2$, $y' = (1/w_1)y$, $x_1' = (1/w_2)x_1$, and $z' = (1/w_2)z$.

Let $\mathbf{1}$ denote an all 1 vector (of appropriate dimension).

Suppose we find a solution to

$$Ax_2' + y' = \mathbf{1} \quad \text{and} \quad Bx_1' + z' = \mathbf{1}$$

$x_1', x_2', y', z' \geq 0$, $(y')^T x_1' = 0$, and $(z')^T x_2' = 0$.

<u>If, in addition,</u> $x_1' \neq 0$ or $x_2' \neq 0$, then, by complementarity <u>both</u> $x_1' \neq 0$ and $x_2' \neq 0$.

In this case we can "recover" a solution x_1, x_2, y, z, and w_1 and w_2 to the original constraints, by multiplying x_1' and x_2' by "normalizing" constants w_1 and w_2, so that each of $x_1 = w_1 x_1'$ and $x_2 = w_2 x_2'$ define probability distributions. These normalizing constants define w_1 and w_2 in our solution.

2-player NE's as a Linear Complementarity Problem

Let

$$M = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \quad u = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} \quad v = \begin{bmatrix} y' \\ z' \end{bmatrix}$$

"**Our Goal:**" Find a solution u, v, to

$$Mu + v = \mathbf{1}$$

such that $u, v \geq 0$, and $u^T v = 0$.

This is an intance of a Linear Complementarity Problem, a classic problem in mathematical programming (see, e.g., the book [Cottle-Pang-Stone'92]).

But, we already know one solution: $u = 0$, $v = \mathbf{1}$.

Our Actual Goal: is to find a solution where $u \neq 0$.

Wait! Doesn't "$Mu + v = \mathbf{1}$" look familiar??

Sure! It's just a "Feasible Dictionary" (from lect. 6 on Simplex), with "Basis" the variables in vector v.

Question: How do we move from this "*complementary basis*" to one where $u \neq 0$?

Answer: Pivoting!! (in a very selective way)

sketch of the Lemke-Howson Algorithm

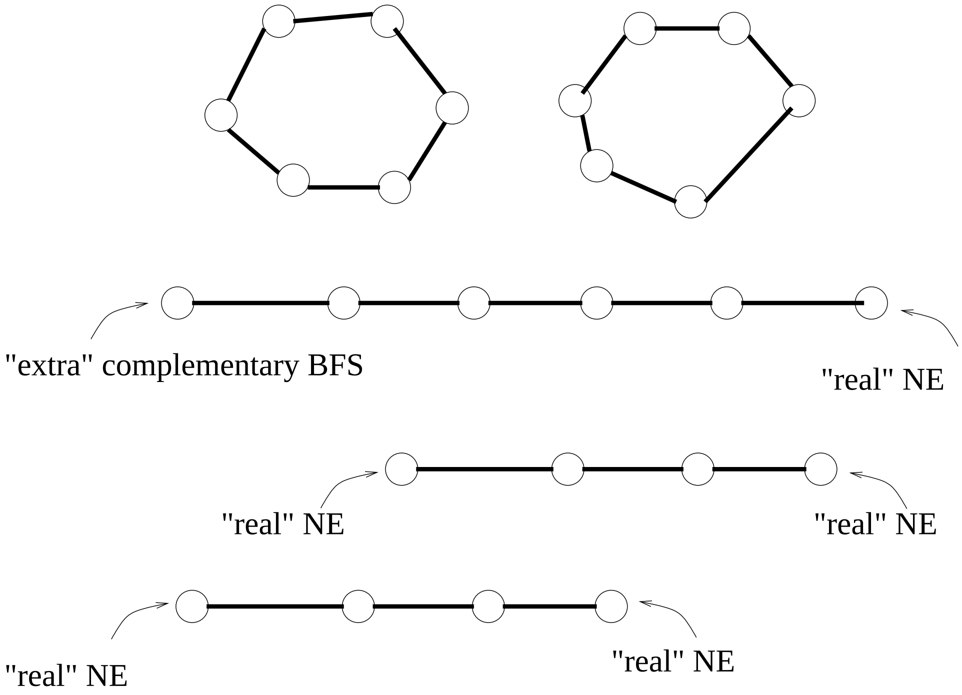
- Start at the "extra" "complementary Basis" $\beta = \{(v)_1, \ldots, (v)_m\}$, where $m = m_1 + m_2$ (with BFS $u = 0, v = \mathbf{1}$). A basis β is **complementary** if for $k \in \{1, \ldots, m\}$, either $(u)_k \notin \beta$ or $(v)_k \notin \beta$ (but not both, since $|\beta| = m$).
- For some i, move via pivoting to a "neighboring" "i-almost complementary" basis β'. A basis β' is **i-almost complementary** if for $k \in \{1, \ldots, m\} \setminus \{i\}$, $(u)_k \notin \beta'$ or $(v)_k \notin \beta'$.
- While (new basis isn't actually complementary)
  - There is a unique j, such that both $(u)_j$ and $(v)_j$ are not in the new basis: one of them was just kicked out of the basis.
  - If $(u)_j$ was just kicked out, move $(v)_j$ into the basis by pivoting. If $(v)_j$ was just kicked, move $(u)_j$ in. (Selective pivot rules assure only one possible entering/leaving pair each iteration.)
  - Newest basis is also i-almost complementary.
- STOP: we have reached a different complementary basis & BFS. A Nash Equilibrium is obtained by "normalizing" $u = [x'_1 \; x'_2]^T$.

We are, of course, skipping lots of details related to “degeneracy”, etc. (similar to complications that arose in Simplex pivoting).

Question Why should this work?

A key reason: With appropriately selective pivoting rules, each i-almost complementary Basis (“vertex”) has 2 neighboring “vertices” unless it is actually a complementary Basis, in which case it has 1. This assures that starting at the “extra” complementary BFS, we will end up at “the other end of the line”.

Let’s see it in pictures:



remarks

- The Lemke-Howson (1964) algorithm has, as you can imagine, a "geometric" interpretation. (See, [von Stengel'2002, HGT, Chap. 45]. Our treatment is closer to [McKelvey-McLennan'96].)
- The algorithm's correctness provides another proof of Nash's theorem, just like Simplex's gives another proof of Minimax (via LP-duality).
- How fast is the LH-algorithm? Unfortunately, examples exist requiring exponentially many pivots, for any permissible pivots (see [Savani-von Stengel'03]).
- Is there a polynomial time algorithm to find a NE in 2-player games? This is a famous open problem. ([Papadimitriou'01] claims it, together with Factoring, to be "the most important open question on the boundary of P today".)
- However, finding "good" NE's that, e.g., maximize "social welfare" is NP-hard. Even knowing whether there is > 1 NE is NP-hard. ([Gilboa-Zemel'89], [Conitzer-Sandholm'03]).

  In practice we may want NE's that optimize some "goodness". The NP-hardness of doing so for many notions of "good", for me diminishes the importance of efficiently finding "any lousy" NE.

games with > 2 players

- Nash himself (1951, page 294) gives a 3 player "poker" game where the only NE is irrational.
  So, it isn't so sensible to speak of computing an "exact" NE when the number of players is > 2.
- But that doesn't stop us from trying to approximate NEs:
  Using the system of non-linear constraints we obtained for algorithm 1, and using very deep algorithms for solving non-linear systems and the "theory of reals" ([. . . ,Canny'88,Renegar'92]) you can show:
  For an n-player finite game Γ, we can approximate an NE to "i-bits of accuracy" in "polynomial space" and exponential time in the size of Γ, and time polynomial in i (given in unary).
  Practical algorithms are another matter . . .
  (see, [McKelvey-Mclennan'96]).
- (Note: $(n + 1)$-player zero-sum games are no easier to solve than n-player general games: any n-player game is "equivalent" to a $(n + 1)$-player zero-sum game: the extra player has only 1 strategy, all payoffs are the same except the $n + 1$'s player's payoff is always the negation of the sum of everyone else's payoffs.)