Algorithmic Game Theory and Applications

Lecture 5: Introduction to Linear Programming

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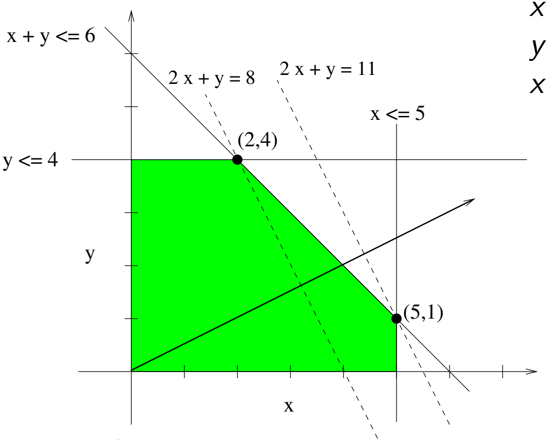
"real world example": the diet problem

- You are a fastidious eater. You want to make sure that every day you get enough of each vitamin: vitamin 1, vitamin 2,...., vitamin m.
- You are also frugal, and want to spend as little as possible.
- There are n foods available to eat: food 1, food 2, ..., food n.
- Each unit of food j has $a_{i,j}$ units of vitamin i.
- Each unit of food j costs c_j.
- Your daily need for vitamin i is b_i units.
- Assume you can buy each food in fractional amounts. (This makes your life much easier.)
- How much of each food would you eat per day in order to have all your daily needs of vitamins, while minimizing your cost?

A Linear Programming Example

Find $(x, y) \in \mathbb{R}^2$ so as to: Maximize $2x + y$

Subject to conditions ("constraints"): $x + y \leq 6;$ $x \leq 5;$ $y \leq 4;$ $x, y \geq 0;$



Much of this simple "geometric intuition" generalizes nicely to higher dimensions. (But be very careful! Things get complicated very quickly!)

The General Linear Program

Definition: A *Linear Programming* or *Linear Optimization* problem instance $(f, \texttt{Opt}, C)$, consists of:

1. A linear *objective function* $f : \mathbb{R}^n \mapsto \mathbb{R}$, given by:
   $f(x_1, \ldots, x_n) = c_1\, x_1 + c_2\, x_2 + \ldots + c_n\, x_n + d$
   where we assume the coefficients c_i and constant d are rational numbers.
2. An optimization criterion: $\texttt{Opt} \in \{\texttt{Maximize}, \texttt{Minimize}\}$.
3. A set (or "system") $C(x_1, \ldots, x_n)$ of m *linear constraints*, or linear inequalities/equalities,
   $C_i(x_1, \ldots, x_n)$, $i = 1, \ldots, m$, where each $C_i(x)$ has form:

$$a_{i,1}\, x_1 + a_{i,2}\, x_2 + \ldots + a_{i,n}\, x_n \ \Delta \ b_i$$

where $\Delta \in \{\leq, \geq, =\}$, and where $a_{i,j}$'s and b_i's are rational numbers.

What does it mean to solve an LP?

For a constraint $C_i(x_1, \ldots, x_n)$, we say vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ *satisfies* $C_i(x)$ if, plugging in v for the variables $x = (x_1, \ldots, x_n)$, the constraint $C_i(v)$ holds true.

For example, $(3, 6)$ satisfies $-x_1 + x_2 \leq 7$.

$v \in \mathbb{R}^n$ is called a *solution* to a system $C(x)$, if v satisfies every constraint $C_i \in C$. I.e., $C_1(v) \wedge \ldots \wedge C_m(v)$ is true.

Let $K(C) \subseteq \mathbb{R}^n$ denote the set of all solutions to the system $C(x)$. We say C is **feasible** if $K(C)$ is not empty.
An *optimal solution*, for $\texttt{Opt} = \texttt{Maximize}$, is some $x^* \in K(C)$ such that:

$$f(x^*) = \max_{x \in K(C)} f(x)$$

(respectively, $f(x^*) = \min_{x \in K(C)} f(x)$, for $\texttt{Opt} = \texttt{Minimize}$)).
Given an LP problem $(f, \texttt{Opt}, C)$, our goal in principle is to find an "optimal solution".

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Given an LP problem $(f, \texttt{Opt}, C)$, our goal in principle is to find an "optimal solution". **Oops!!** There may not be an optimal solution!

Things that can go wrong

Two things can go wrong when looking for an optimal solution:

1. There may be no solutions at all!
   I.e., C is not feasible, i.e., $K(C)$ is empty. Consider:
   Maximize x
   Subject to: $x \leq 3$ *and* $x \geq 5$

2. $\max / \min_{x \in K(C)} \; f(x)$ may not exist (!), because $f(x)$ is unbounded above/below in $K(C)$. Consider:
   Maximize x
   Subject to: $x \geq 5$

So, we have to revise our goals to handle these cases.

Note: If we allowed strict inequalities, e.g., $x < 5$, there would have been yet another problem:

Maximize x
Subject to: $x < 5$

The LP Problem Statement

Given an LP problem instance $(f, \texttt{Opt}, C)$ as input, output one of the following three:

1. "The problem is Infeasible."
2. "The problem is Feasible But Unbounded."
3. "An Optimal Feasible Solution (OFS) exists.
   One such optimal solution is $x^* \in \mathbb{R}^n$.
   The optimal objective value is $f(x^*) \in \mathbb{R}$."

Oops!! It seems yet another thing could go wrong: What if every optimal solution $x^* \in \mathbb{R}^n$ is irrational?
How can we "output" irrational numbers?
Likewise, what if the Opt value $f(x^*)$ is irrational?

Fact: This problem never arises. The above three answers cover all possibilities, and furthermore, as long as all our coefficients and constants are rational, if an OFS exists, a rational OFS x^* exists, and the optimal value $f(x^*)$ is also rational. (We will learn why later.)

Simplified forms for LP problems

1. In principle, we need only consider Maximization, because:
$$\min_{x \in K} f(x) = -\max_{x \in K} -f(x)$$
(either side is unbounded if and only if both are.)
2. We only need an objective function $f(x_1, \ldots, x_n) = x_i$, for some x_i, because we can:
Introduce new variable x_0. Add new constraint $f(x) = x_0$ to constraints C. Make the new objective "Optimize x_0".
3. Don't need "$=$" constraints: $\alpha = \beta \Leftrightarrow (\alpha \leq \beta \wedge \alpha \geq \beta)$.
4. Don't need "$\alpha \geq b$", where $b \in \mathbb{R}$: $\alpha \geq b \Leftrightarrow -\alpha \leq -b$.
5. We can constrain every variable x_i to be $x_i \geq 0$:
Introduce two variables x_i^+, x_i^- for each variable x_i.
Replace each occurence of x_i by $(x_i^+ - x_i^-)$, and add the constraints $x_i^+ \geq 0$, $x_i^- \geq 0$.
(**N.B.** can't do both (2.) and (5.) together.)

A lovely but terribly inefficient algorithm for LP

Input: LP instance $(x_0, \texttt{Opt}, C(x_0, x_1, \ldots, x_n))$.

1. **For** $i = n$ downto 1
   a. Rewrite each constraint involving x_i as $\alpha \leq x_i$, or as $x_i \leq \beta$. (One of the two is possible.) Let these be:
   $\alpha_1 \leq x_i, \ldots, \alpha_k \leq x_i$; $x_i \leq \beta_1, \ldots, x_i \leq \beta_r$
   (Retain these constraints, H_i, for later.)
   b. Remove H_i, i.e., all constraints involving x_i. Replace with constraints: $\{\alpha_j \leq \beta_l \mid j = 1, \ldots, k, \& l = 1, \ldots, r\}$.
2. Only x_0 (or no variable) remains. All constraints have the forms $a_j \leq x_0$, $x_0 \leq b_l$, or $a_j \leq b_l$, where a_j's and b_l's are constants. It's easy to check "feasibility" & "boundedness" for such a one(or zero)-variable LP, and to find an optimal x_0^* if one exists.
3. Once you have x_0^*, plug it into H_1. Solve for x_1^*. Then use x_0^*, x_1^* in H_2 to solve for x_2^*, ..., use $x_0^*, \ldots, x_{i-1}^*$ in H_i to solve for x_i^*. ... then $x^* = (x_0^*, \ldots, x_n^*)$ is an optimal feasible solution.

remarks on the lovely algorithm

- This algorithm was first discovered by Fourier (1826). Rediscovered in 1900's, by Motzkin (1936) and others.
- It is called *Fourier-Motzkin Elimination*, and can be viewed as a generalization of *Gaussian Elimination*, used for solving systems of linear equalities.
- Why is Fourier-Motzkin so inefficient? In the worst case, if every variable x_i is involved in every constraint, with half of them in each "direction", then each "for loop" interation roughly squares the number of constraints. So, toward the end we could have $\sim \frac{m^{2^n}}{2^n}$ constraints!
- Let's recall Gaussian Elimination (GE). It is much nicer and does not suffer from this explosion.
- In 1947, Dantzig invented the celebrated **Simplex Algorithm** for LP. It can be viewed as a much more refined generalization of GE. Next time, Simplex!

more remarks

Immediate Corollaries of Fourier-Motzkin:

Corollary 1: The three possible "answers" to an LP problem do cover all possibilities.
(In particular, unlike "Maximize x; $x < 5$", If an LP has a "Supremum" it has a "Maximum".)

Corollary 2: If an LP has an OFS, then it has a rational OFS, x^*, and $f(x^*)$ is also rational.
Proof: We used only addition, multiplication, & division by rationals to arrive at the solution. □

further remarks

Although Fourier-Motzkin is bad in the worst case, it can still be quite useful. It can be used to remove redundant variables and constraints[1]. And its worst-case behavior may in many cases not arise in practice.

Generalizations of Fourier-Motzkin are used in some tools (e.g., [Pugh,'92]) for solving "Integer Linear Programming", where we seek an optimal solution x^* not in $\mathbb{R}^n$, but in $\mathbb{Z}^n$. ILP is a **much harder** problem! (**NP**-complete.)

For ordinary LP however, Fourier-Motzkin can't compete with Simplex.

[1]When a variable x_i is only involved in inequalities, all in one "direction", those inequality constraints are all "redundant" because they can always be satisfied by setting x_i to a sufficiently high/low value.

- **Food for Thought:** Think about what kinds of clever heuristics and hacks you could use during Fourier-Motzkin to keep the number of constraints as small as possible. E.g., In what order would you try to eliminate variables? (Clearly, any order is fine, as long as x_0 is last.)