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# Algorithmic Game Theory and Applications

## Lecture 3: Nash's Theorem

Kousha Etessami



# The Brouwer Fixed Point Theorem

We will use the following to prove Nash's Theorem.

**Theorem**(Brouwer, 1909) Every continuous function  $f : D \mapsto D$  mapping a compact and convex, nonempty subset  $D \subseteq \mathbb{R}^m$  to itself has a “fixed point”, i.e., there is  $x^* \in D$  such that  $f(x^*) = x^*$ .

Explanation:

- A “continuous” function is intuitively one whose graph has no “jumps”. I.e., any “sufficiently little (non-zero) change” in  $x$  can change  $f(x)$  by at most “as little (non-zero) change as desired”.
- For our current purposes, we don't need to know exactly what “compact and convex” means.

(See the appendix of this lecture for definitions.)

We only state the following fact:

**Fact** The set of profiles  $X = X_1 \times \dots \times X_n$  is a compact and convex subset of  $R^m$ .

(Where  $m = \sum_{i=1}^n m_i$ , recalling that  $m_i = |S_i|$ .)

# Simple cases of Brouwer's Theorem

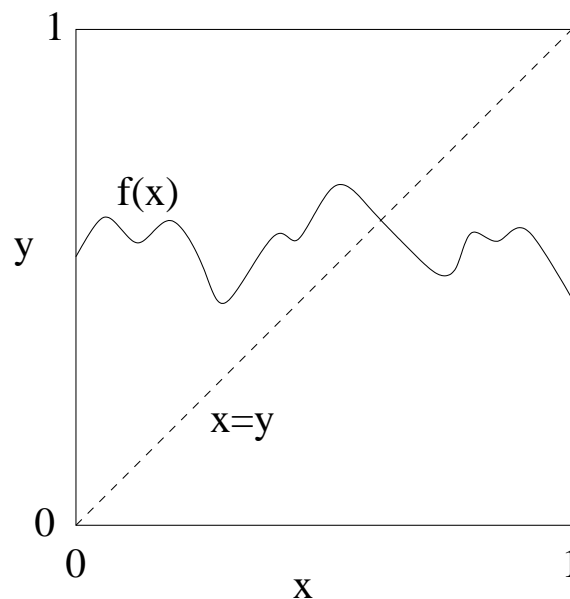
To see a simple example of what Brouwer's theorem says, consider the interval  $[0, 1] = \{x \mid 0 \leq x \leq 1\}$ .

$[0, 1]$  is compact and convex.

(More generally,  $[0, 1]^n$  is compact and convex.)

For a continuous  $f : [0, 1] \mapsto [0, 1]$ , you can “visualize” why the theorem is true:

The “visual proof” in the 1-dimensional case:



For  $f : [0, 1]^2 \mapsto [0, 1]^2$ , the theorem is already far less obvious: “the crumpled sheet experiment”.

## brief remarks

- Brouwer's Theorem is a deep and important result in topology.
- It is not easy to prove, and we won't prove it.
- If you are desperate to see a proof, there are many. See, e.g., any of these:
  - [Milnor'66] (Differential Topology). (uses, e.g., Sard's Theorem).
  - [Scarf'73, Border'89], with Economics viewpoints (they use Sperner's Lemma).
  - [Rotman'88] (Algebraic Topology). (uses homology, etc.)
  - [Papadimitriou's Berkeley Lecture Notes '03] (uses Sperner's Lemma).
  - Possibly my favorite proof:  
[D. Gale'79] , uses the fact that HEX (a finite, extensive form game of perfect information, re-invented by Nash) is a “win-lose” game, i.e., any n-dimensional Hex game has a winner (i.e., can not end in a draw).

## proof of Nash's theorem

**Proof:** (Nash's 1951 proof)

We will define a continuous function  $f : X \mapsto X$ , where  $X = X_1 \times \dots \times X_n$ , and we will show that if  $f(x^*) = x^*$  then  $x^* = (x_1^*, \dots, x_n^*)$  must be a Nash Equilibrium.

By Brouwer's Theorem, we will be done.

(In fact, it will turn out that  $x^*$  is a Nash Equilibrium if and only if  $f(x^*) = x^*$ .)

We start with a claim.

**Claim:** A profile  $x^* = (x_1^*, \dots, x_n^*) \in X$  is a Nash Equilibrium if and only if, for every player  $i$ , and every pure strategy  $\pi_{i,j}$ ,  $j \in S_i$ ,

$$U_i(x^*) \geq U_i(x_{-i}^*; \pi_{i,j})$$

**proof of claim:** If  $x^*$  is a NE then, it is obvious by definition that  $U_i(x^*) \geq U_i(x_{-i}^*; \pi_{i,j})$ .

For the other direction: by calculation it is easy to see that for any mixed strategy  $x_i \in X_i$ ,

$$U_i(x_{-i}^*; x_i) = \sum_{j=1}^{m_i} x_i(j) * U_i(x_{-i}^*; \pi_{i,j})$$

I.e., the payoff of Player  $i$  is the “weighted average” of the payoffs of each of its pure strategies,  $j$ , weighted by the probability  $x_i(j)$  of that strategy.

By assumption,  $U_i(x^*) \geq U_i(x_{-i}^*; \pi_{i,j})$ , for all  $j$ .

So, clearly  $U_i(x^*) \geq U_i(x_{-i}^*; x_i)$ , for any  $x_i \in X_i$ , because a “weighted average” of things no bigger than  $U_i(x^*)$  can't be bigger than  $U_i(x^*)$ .

Hence, each  $x_i^*$  is a best response strategy to  $x_{-i}^*$ . In other words,  $x^*$  is a Nash Equilibrium. ■

So, rephrasing our goal, we want to find  $x^* = (x_1^*, \dots, x_n^*)$  such that

$$U_i(x_{-i}^*; \pi_{i,j}) \leq U_i(x^*)$$

i.e., such that

$$U_i(x_{-i}^*; \pi_{i,j}) - U_i(x^*) \leq 0$$

for all players  $i \in N$ , and all  $j = 1, \dots, m_i$ .

For a mixed profile  $x = (x_1, x_2, \dots, x_n) \in X$ : let

$$\varphi_{i,j}(x) = \max\{0, U_i(x_{-i}; \pi_{i,j}) - U_i(x)\}$$

Intuitively,  $\varphi_{i,j}(x)$  measures “how much better off” player  $i$  would be if he/she picked  $\pi_{i,j}$  instead of  $x_i$  (and everyone else remained unchanged).

Define  $f : X \mapsto X$  as follows: For  $x = (x_1, x_2, \dots, x_n) \in X$ , let

$$f(x) = (x'_1, x'_2, \dots, x'_n)$$

where for all  $i$ , and  $j = 1, \dots, m_i$ ,

$$x'_i(j) = \frac{x_i(j) + \varphi_{i,j}(x)}{1 + \sum_{k=1}^{m_i} \varphi_{i,k}(x)}$$

**Facts:**

1. If  $x \in X$ , then  $f(x) = (x'_1, \dots, x'_n) \in X$ .
2.  $f : X \mapsto X$  is continuous.

(These facts are not hard to check.)

Thus, by Brouwer, there exists  $x^* = (x_1^*, x_2^*, \dots, x_n^*) \in X$  such that  $f(x^*) = x^*$ .

Now we have to show  $x^*$  is a NE.

For each  $i$ , and for  $j = 1, \dots, m_i$ ,

$$x_i^*(j) = \frac{x_i^*(j) + \varphi_{i,j}(x^*)}{1 + \sum_{k=1}^{m_i} \varphi_{i,k}(x^*)}$$

thus,

$$x_i^*(j) \left(1 + \sum_{k=1}^{m_i} \varphi_{i,k}(x^*)\right) = x_i^*(j) + \varphi_{i,j}(x^*)$$

hence,

$$x_i^*(j) \sum_{k=1}^{m_i} \varphi_{i,k}(x^*) = \varphi_{i,j}(x^*)$$

We will show that in fact this implies  $\varphi_{i,j}(x^*)$  must be equal to 0 for all  $j$ .

**Claim:** For any mixed profile  $x$ , for each player  $i$ , there is some  $j$  such that  $x_i(j) > 0$  and  $\varphi_{i,j}(x) = 0$ .

Proof of claim: For any  $x \in X$ ,

$$\varphi_{i,j}(x) = \max\{0, U_i(x_{-i}; \pi_{i,j}) - U_i(x)\}$$

Since  $U_i(x)$  is the “weighted average” of  $U_i(x_{-i}; \pi_{i,j})$ 's, based on the “weights” in  $x_i$ , there must be some  $j$  used in  $x_i$ , i.e., with  $x_i(j) > 0$ , such that  $U_i(x_{-i}; \pi_{i,j})$  is no more than the weighted average. I.e.,

$$U_i(x_{-i}; \pi_{i,j}) \leq U_i(x)$$

I.e.,

$$U_i(x_{-i}; \pi_{i,j}) - U_i(x) \leq 0$$

Therefore,

$$\varphi_{i,j}(x) = \max\{0, U_i(x_{-i}; \pi_{i,j}) - U_i(x)\} = 0$$

■

Thus, for such a  $j$ ,  $x_i^*(j) > 0$  and

$$x_i^*(j) \sum_{k=1}^{m_1} \varphi_{i,k}(x^*) = 0 = \varphi_{i,j}(x^*)$$

But, since  $\varphi_{i,k}(x^*)$ 's are all  $\geq 0$ , this means  $\varphi_{i,k}(x^*) = 0$  for all  $k = 1, \dots, m_i$ . Thus,

For all players  $i$ , and for  $j = 1, \dots, m_i$ ,

$$U_i(x^*) \geq U_i(x_{-i}^*; \pi_{i,j})$$

### **Q.E.D. (Nash's Theorem)**

In fact, since  $U_i(x^*)$  is the “weighted average” of  $U_i(x_{-i}^*; \pi_{i,j})$ 's, we see that

### **Useful Corollary for Nash Equilibria:**

$U_i(x^*) = U_i(x_{-i}^*; \pi_{i,j})$ , whenever  $x_i^*(j) > 0$ .

Rephrased: In a Nash Equilibrium  $x^*$ , if  $x_i^*(j) > 0$  then  $U_i(x_{-i}^*; \pi_{i,j}) = U_i(x^*)$ ; i.e., each such  $\pi_{i,j}$  is itself a “best response” to  $x_{-i}^*$ .

This is a subtle but very important point.

It will be useful later when we try to compute NE's.

## Remarks

- The proof using Brouwer gives ostensibly no clue how to compute a Nash Equilibrium. It just says it exists!
- We will come back to the question of computing Nash Equilibria in general games later in the course.
- We start next time with a special case: 2-player zero-sum games (e.g., of the Rock-Paper-Scissor's variety). These have an elegant theory (von Neumann 1928), predating Nash.
- To compute solutions for 2p-zero-sum games, Linear Programming will come into play. Linear Programming is a very important tool in algorithms and optimization. Its uses go FAR beyond solving zero-sum games. So it will be a good opportunity to learn about LP.

## pleasure reading: evolutionary biology as a game

- One way to view how we might “arrive” at a Nash equilibrium is through a process of evolution.
- John Maynard Smith (1972-3,'82) introduced game theoretic ideas into evolutionary biology with the concept of an Evolutionarily Stable Strategy.
- Your pleasure reading is from Straffin(1993) which gives a very amusing introduction to this.
- Intuitively, a mixed strategy can be viewed as percentages in a population that exhibit different behaviors (strategies).
- Their behaviors effect each other’s survival, and thus each strategy has a certain survival value dependent on the strategy of others.
- The population is in “evolutionary equilibrium” if no “mutant” strategy could invade it and “take over” .

## a glossary for your reading

- **Definition** A 2-player game is **symmetric** if  $S_1 = S_2$ , and for all  $s_1, s_2 \in S_1$ ,  $u_1(s_1, s_2) = u_2(s_2, s_1)$ .
- **Definition** In a 2p-sym-game, mixed strategy  $x_1^*$  is an **Evolutionarily Stable Strategy (ESS)**, if:
  1.  $x_1^*$  is a best response to itself, i.e.,  $x^* = (x_1^*, x_1^*)$  is a symmetric Nash Equilibrium, &
  2. If  $x_1' \neq x_1^*$  is any other best response to  $x_1^*$ , then  $U_1(x_1', x_1') < U_1(x_1^*, x_1')$ .

Nash (1951, p. 289) also proves that every symmetric game has a symmetric NE,  $(x_1^*, x_1^*)$ . (However, not every symmetric game has a ESS.)

- Given a profile  $x \in X$  in an  $n$ -player game, the **“(purely utilitarian) social welfare”** is:

$$U_1(x) + U_2(x) + \dots + U_n(x).$$

- A profile  $x \in X$  is **pareto efficient** (a.k.a., **pareto optimal**) if there is no other profile  $x'$  such that  $U_i(x) \leq U_i(x')$  for all players  $i$ , and such that for some player  $k$ ,  $U_k(x) < U_k(x')$ .
- (Prisoner’s Dilemma shows that NE’s need not optimize social welfare nor be Pareto optimal.)

## Appendix: continuity, compactness, convexity

**Definition** For  $x, y \in \mathbb{R}^n$ ,  $\text{dist}(x, y) = \sqrt{\sum_{i=1}^n (x(i) - y(i))^2}$  denotes the Euclidean distance between points  $x$  and  $y$ .

A function  $f : D \subseteq \mathbb{R}^n \mapsto \mathbb{R}^n$  is **continuous at a point**  $x \in D$  if for all  $\epsilon > 0$ , there exists  $\delta > 0$ , such that for all  $y \in D$ : if  $\text{dist}(x, y) < \delta$  then  $\text{dist}(f(x), f(y)) < \epsilon$ .

$f$  is called **continuous** if it is continuous at every point  $x \in D$ .

**Definition** A set  $K \subseteq \mathbb{R}^n$  is **convex** if for all  $x, y \in K$  and all  $\lambda \in [0, 1]$ ,  $\lambda x + (1 - \lambda)y \in K$ .

Rather than stating a general definition of compactness for arbitrary topological spaces, we use the following fact as a definition, restricted to Euclidean space:

**Fact** A set  $K \subseteq \mathbb{R}^n$  is **compact** if and only if it is **closed** and **bounded**. (So, we need to define “closed” and “bounded”.)

**Definition** A set  $K \subseteq \mathbb{R}^n$  is **bounded** iff there is some non-negative integer  $M$ , such that  $K \subseteq [-M, M]^n$ . (i.e.,  $K$  “fits inside” a finite  $n$ -dimensional box.)

**Definition** A set  $K \subseteq \mathbb{R}^n$  is **closed** iff for all sequences  $x_0, x_1, x_2, \dots$ , where for all  $i \geq 0$ ,  $x_i \in K$ , if there exists  $x \in \mathbb{R}^n$  such that  $x = \lim_{i \rightarrow \infty} x_i$  (i.e., for all  $\epsilon > 0$ , there exists integer  $k > 0$  such that  $\text{dist}(x, x_m) < \epsilon$  for all  $m > k$ ), then  $x \in K$ .

(In other words, if a sequence of points is in  $K$  then its limit (if it exists) must also be in  $K$ .)