Algorithmic Game Theory and Applications

Lecture 17:
A first look at Auctions and Mechanism Design:
Auctions as Games, Bayesian Games, Vickrey auctions

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Food for thought: sponsored search auctions

Question

How should advertisement slots on your Google search page be auctioned?

You may have already heard that Google uses a so-called "Generalized Second Price Auction" mechanism to do this.

But why do they do so? Is there any "better" way?
What does "better" actually mean? What is the goal of the auctioneer?
How should other (electronic) auctions be conducted?

More generally, how should we design games when our goal is to compel selfish players to behave in a desired way (e.g., a "socially optimal" way, or "truthfully"). This is the topic of Mechanism Design.

Auction theory and Mechanism Design are rich and vast subdisciplines of game/economic theory. We can not do them adequate justice. We briefly explore them in remaining lectures, focusing on algorithmic aspects.
Some reference reading: Chapters 9, 11, 12, 13, & 28 of AGT book.

Auctions as games

Consider one formulation of a single-item, sealed-bid, auction as a game:

- Each of n bidders is a player.
- Each player i has a valuation, $v_i \in \mathbb{R}$, for the item being auctioned.
- if the outcome is: player i wins the item and pays price pr, then the payoff to player i, is

$$u_i(outcome) := v_i - pr$$

  and all other players $j \neq i$ get payoff 0: $u_j(outcome) := 0$.
- Let us require that the auctioneer must set up the rules of the auction so that they satisfy the following reasonable constraints: given (sealed) bids $(b_1, \ldots, b_n)$, one of the highest bidders must win, and must pay a price pr such that $0 \leq pr \leq \max_i b_i$.
- Question: What rule should the auctioneer employ, so that for each player i, bidding their "true valuation" v_i (i.e., letting $b_i := v_i$) is a dominant strategy?

Vickrey auctions

- In a Vickrey auction, a.k.a., *second-price, sealed bid* auction, a highest bidder, j, whose bid is $b_j = \max_i b_i$, gets the item, but pays the second highest bid price: $pr = \max_{i \neq j} b_i$.

Claim

Bidding their true valuation, v_i, i.e., letting $b_i := v_i$, is a (weakly) dominant strategy in this game for all players i.

Let us prove this on the board. (We actually prove something stronger.)

- Note: there is *something very fishy/unsatisfactory* about our formulation so far of an auction as a *complete information* game: player i normally does not know the valuation v_j of other players $j \neq i$. But if viewed as a *complete information* game, then every player knows every one else's valuation. This is totally unrealistic!
- We need a better game-theoretic model for settings like auctions.

Bayesian Games (Games of Incomplete Information) [Harsanyi,'67,'68]

A **Bayesian Game**, $G = (N,\ (A_i)_{i\in N},\ (T_i)_{i\in N},\ (u_i)_{i\in N},\ p)$, has:

- A (finite) set $N = \{1, \ldots, n\}$ of **players**.
- A (finite[1]) set A_i of **actions** for each player $i \in N$.
- A (finite[1]) set of **possible types**, T_i, for each player $i \in N$.
- A **payoff (utility) function**, for each player $i \in N$:

$$u_i : A_1 \times \ldots \times A_n \times T_1 \times \ldots \times T_n \to \mathbb{R}$$

- A **(joint) probability distribution over types**:

$$p : T_1 \times \ldots \times T_n \to [0,1]$$

where, letting $T = T_1 \times \ldots \times T_n$, we must have:

$$\sum_{(t_1,\ldots,t_n)\in T} p(t_1, \ldots, t_n) = 1$$

p is sometimes called a **common prior**.

[1] We can and do often remove the finiteness assumption on the type/action spaces, e.g., letting T_i be a closed interval $[a,b] \subseteq \mathbb{R}$.

strategies and expected payoffs in Bayesian games

- A pure strategy for player i is a function $s_i : T_i \rightarrow A_i$.
  I.e., player i knows its own type, t_i, and chooses action $s_i(t_i) \in A_i$.
  (In a mixed strategy, x_i, $x_i(t_i)$ is a probability distribution over A_i.)
- Players' types are chosen randomly according to (joint) distribution p.
- Player i knows $t_i \in T_i$, but doesn't know the type t_j of players $j \neq i$.
- But every player "knows" the joint distribution p (this is often too strong an assumption, dubiously so, but it is useful), so each player i can compute the conditional probabilities, $p(t_{-i} \mid t_i)$, on other player's types, given its own type t_i.
- The **expected payoff** to player i, under the pure profile $s = (s_1, \ldots, s_n)$, when player i has type t_i (which it knows) is:

$$U_i(s, t_i) = \sum_{t_{-i}} p(t_{-i} \mid t_i) u_i(s_1(t_1), \ldots, s_n(t_n), t_i, t_{-i})$$

Bayesian Nash Equilibrium

Definition

A strategy profile $s = (s_1, \ldots, s_n)$ is a (pure) Bayesian Nash equilibrium (BNE) if for all players i and all types $t_i \in T_i$, and all strategies s'_i for player i, we have: $U_i(s, t_i) \geq U_i((s'_i; s_{-i}), t_i)$.
(A **mixed** *BNE is defined similarly, by allowing players to use mixed strategies, x_i, and defining expected payoffs $U_i(x, t_i)$ accordingly.)*

Proposition

Every finite Bayesian Game has a mixed strategy BNE.

Proof

Follows from Nash's Theorem. Every finite Bayesian Games can be encoded as a finite extensive form game of imperfect information: the game tree first randomly choose a subgame labeled $(t_1, \ldots, t_n)$, with probability $p(t_1, \ldots, t_n)$. All nodes belonging to player i in different subgames labeled by the same type $t_i \in T_i$ are in the same information set.

Back to Vickrey auctions

Now suppose we model a sealed-bid single-item auction using a Bayesian game, with some arbitrary prior probability distribution $p(v_1, \ldots, v_n)$ over valuations (suppose every v_i is in some finite nonnegative range $[0, v_{max}]$). The private information of each player i is $t_i := v_i$.

Proposition

In the Vickrey (second-price, sealed-bid) auction game, with any prior p, the truth revealing profile of bids $v = (v_1, \ldots, v_n)$, is a weakly dominant strategy profile.
(In fact, under suitable conditions on p, v is the unique BNE of the game.)

Proof

The exact same proof as for the complete information version of the vickrey game works to show that v is a weakly dominant strategy profile. Indeed, we didn't use the fact that the game has complete information. We didn't even assume the players have a common prior for this!

What about other auctions?

- In first-price, sealed-bid auctions, (maximum bidder gets the item and pays $pr = \max_i b_i$), bidding truthfully may indeed not be a dominant strategy. E.g., if player i knows (with high probability) that its valuation $v_i >> v_j$ for all $j \neq i$, then i will not "waste money" and will instead bid $b_i << v_i$, since it will still get the item.
- What implications does this have for the expected revenue of the auction? Surprisingly, it has less than you might think:

Proposition

If prior p is a product of i.i.d. uniform distributions over some interval $[0, v_{max}]$, then the expected revenue of the second-price and first-price sealed-bid auctions are both the same in their (unique) symmetric BNEs.

This is actually a special case of a much more general result in Mechanism Design called the Revenue Equivalence Principle. (But anyway, note that one-shot revenue maximization is not always a wise goal for an auctioneer.)

The bigger picture

- This was our first look at Auctions (with hints of Mechanism design), in the simple setting of a single-item, sealed-bid, auction.
- To delve deeper into (algorithmic aspects of) Mechanism Design and auctions, we need a deeper understanding of the "bigger picture":
- Starting next time, we will step back to glimpse at where all this fits in the broader scope of Economic Theory.
- In particular, we will briefly discuss some social choice theory, and also Market equilibria, before returning to the important VCG mechanism, which vastly generalizes the Vickrey single-item auction.
- Some of the topics we will discuss are beyond the scope of this AGT course, and can be properly learned, e.g., in a good Microeconomic Theory course/text (e.g., [Mas-Colell-Whinston-Green'95]).
- However, I feel I would be "cheating you" if I didn't at least provide some (necessarily impressionistic) sense of the "bigger picture".