Algorithmic Game Theory
and Applications

Lecture 16:
Selfish Network Routing,
Congestion Games,
and the Price of Anarchy

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**warning, again**

- In the few remaining lectures, we will briefly look at several **vast** subjects at the forefront of AGT research, at the intersection of CS, economics and e-commerce, and “internetology”.

- These are extremely active research topics, and there are entire conferences and workshops dedicated to them. See, e.g., the *ACM conference on Economics and Computation*.


  For today’s lecture, Chapter 18 (Roughgarden) and Chapter 19 (Tardos and Wexler), are relevant.

- We will only scratch the surface of these subjects, so please explore them further if they interest you.
games and the internet

• Basic idea: “The internet is a huge experiment in interaction between agents (both human and automated).”

• Such interactions can profitably be viewed from a game theoretic viewpoint: agents trying to maximize their own payoffs, etc.

• What are the implications of selfish behavior?

• How do we set up the rules of these games to harness “socially optimal” results?
(Selfish) Network Routing as a Game

Figure 1: “The Internet”

- Selfish agent $i = 1, 2, 3$, wants to route packets from source $S_i$ to destination $T_i$. So, agent $i$ must choose a directed path from $S_i$ to $T_i$.
- The delay on each edge of the path is governed by the congestion of that edge, i.e., by the total number of agents using that edge in their path.
- Agents may want to change their choice if their path is too congested.
- What is a Nash Equilibrium in this game?
- What are the welfare properties of such NE? (Is it socially optimal? If not, how bad can it be?)
A Congestion Game, \( G = (N, R, (Z_i)_{i \in N}, (d_r)_{r \in R}) \) has:

- A finite set \( N = \{1, \ldots, n\} \) of players.
- A finite set of \( R = \{1, \ldots, m\} \) of resources.
- For each player, \( i \), a set \( Z_i \subseteq 2^R \), of admissible strategies for player \( i \). So a pure strategy, \( s_i \in Z_i \) is simply a set of resources.
- Each resource \( r \in R \) has a cost function: 
  \[
d_r : \mathbb{N} \rightarrow \mathbb{Z}
\]

Intuitively, \( d_r(j) \) is the cost of using resource \( r \) if there are \( j \) agents simultaneously using \( r \).
- For a pure strategy profile \( s = (s_1, \ldots, s_n) \in Z_1 \times \ldots Z_n \), the congestion on resource \( r \) is:
  \[
n_r(s) = |\{i \mid r \in s_i\}|
\]
- Under strategy profile \( s = (s_1, \ldots, s_n) \), the total cost to player \( i \) is:
  \[
  C_i(s) = \sum_{r \in s_i} d_r(n_r(s))
  \]
- Every player, \( i \), of course want to minimize its own (expected) cost.
Best response dynamics, and pure Nash Equilibria

In a congestion game $G$, for any pure strategy profile $s = (s_1, \ldots, s_n)$, suppose that some player $i$ has a better alternative strategy, $s'_i \in Z_i$, such that $C_i(s_{-i}; s'_i) < C_i(s)$.

Player $i$ can switch (unilaterally) from $s_i$ to $s'_i$. This takes us from profile $s$ to profile $(s_{-i}, s'_i)$.

We call this a single (strict) improvement step.

Starting at an arbitrary pure strategy profile $s$, what happens if the players perform a sequence of such improvement steps?

**Theorem:** ([Rosenthal’73]) *In any congestion game, every sequence of strict improvement steps is necessarily finite, and terminates in a pure Nash Equilibrium.*

*Thus, in particular, every congestion game has a pure strategy Nash Equilibrium.*
Proof: Potential functions

Proof: Consider the following potential function:

\[ \varphi(s) = \sum_{r \in R} \sum_{i=1}^{n_r(s)} d_r(i) \]  

(1)

What happens to the value of \( \varphi(s) \) if player \( i \) switches unilaterally from \( s_i \) to \( s_i' \), taking us from profile \( s \) to profile \( s' := (s_{-i}; s_i') \)?

Claim: \( \varphi(s) - \varphi(s') = C_i(s) - C_i(s') \).

Proof: Re-order the players in any arbitrary way, and index them as players 1, 2, \ldots, n. (In particular, a player formerly indexed \( i \) could be re-indexed as \( n \).)

Then, for \( i' \in \{1, \ldots, n\} \), define

\[ n_r^{(i')}(s) = |\{i \mid r \in s_i \land i \in \{1, \ldots, i'\}\}| \]

By exchanging the order of summation in equation (1) for \( \varphi(s) \), it can be seen that (check this yourself):

\[ \varphi(s) = \sum_{i=1}^{n} \sum_{r \in s_i} d_r(n_r^{(i)}(s)) \]
Now note that $n_r^{(n)}(s) = n_r(s)$. Thus

$$\sum_{r \in s_n} d_r(n_r^{(n)}(s)) = \sum_{r \in s_n} d_r(n_r(s)) = C_n(s)$$

So, if player $n$ switches from strategy $s_n$ to $s'_n$, leading us from profile $s$ to $s' = (s_{-n}; s'_n)$, then:

$$\varphi(s) - \varphi(s') = C_n(s) - C_n(s').$$

But note that when re-ordering we could have chosen player $n$ to be any player we want! So this holds for every player $i$. \qed

To complete the proof of the Theorem:

Observe that every strict improvement step must decreases the value of the potential function $\varphi(s)$ by at least 1 (the costs $d_r(s)$ are all integers).

Furthermore, there are only finitely many pure strategies $s$, so there are finite integers:

$a = \min_s \varphi(s)$ and $b = \max_s \varphi(s)$.

Thus, every improvement sequence is finite.

Finally, note that the last profile $s$ in any improvement sequence which can not be further improved is, by definition, a pure Nash equilibrium. \qed
The complexity of finding a pure NE in network congestion games

Consider a network congestion game where we are given a network with source-sink node pairs \((S_i, T_i)\), for each player \(i\), and each player must to choose a route (path) from \(S_i\) to \(T_i\).

Suppose the cost (delay) of an edge, \(e\), under profile \(s\), is defined to be some linear function:

\[
d_e(n_e(s)) = \alpha_e n_e(s) + \beta_e
\]

One obvious way to compute a pure NE is to perform an arbitrary improvement sequence. However, this may conceivably require many improvement steps.

Is there a better algorithm?

It turns out that it is as hard as any polynomial local search problem to compute a pure NE for network congestion games:

**Theorem:** [Fabrikant et.al.’04, Ackermann et.al.’06].

*Computing a pure NE for a network congestion game is PLS-complete,*

*even when all edge delay functions, \(d_e\), are linear.*

So, unfortunately, a P-time algorithm is unlikely.
A flow network game

A large number $n$ of customers in the network want to go from $s$ to $t$.

Each can either take the edge labeled 1, with “latency” 1 (delay of crossing edge), or edge labeled with latency $x$. Here $x$ represents the “congestion”, given by the ratio of the number of customers that are using that edge divided by the total $n$.

Assume $n$ is very large, (basically, $n \to \infty$).

What is the delay in Nash Equilibrium? (NEs in such a setting yield an essentially unique average delay [Beckmann, et. al. '56].)

What is a “globally optimal” customer routing strategy profile that minimizes average delay? What is the globally optimal average delay?
a modified game

- What is the NE, and what is the average delay it induces?
- What is the globally optimal average delay?
a different network

What is the NE, and what is its average delay?

What is a globally optimal strategy profile and optimal average delay?

What if an ambitious “network service provider” wanted to build additional “superfast” lines?
Braess’s paradox

- What is the NE and its average delay?

- What is the globally optimal average delay?
social welfare
and the price of anarchy

Recall that in a strategic game $\Gamma$, we may have
different measures of the “social welfare” $\text{welfare}(x)$
under a particular profile of (mixed) strategies $x \in X$.
For example, “utilitarian” social welfare is
$\text{welfare}(x) := \sum_{i=1}^{n} U_i(x)$. For a game $\Gamma$, let $\text{NE}(\Gamma)$
be the set of NE’s of $\Gamma$.
For our next definition suppose $\text{welfare}(x) > 0$ for all $x \in X$. (In many games we could enforce this by,
e.g., “shifting” all payoffs by an additive factor.)

A version of “the price of anarchy” can be defined as: ([Koutsoupias-Papadimitriou’98])

$$PA(\Gamma) := \frac{\max_{x \in X} \text{welfare}(x)}{\min_{x \in \text{NE}(\Gamma)} \text{welfare}(x)}$$

Thus, the “price of anarchy” is the ratio of best
“global” outcome to the the worst NE outcome.
Note: this ratio is $\geq 1$ and larger means “worse”.

It would be comforting to establish that in various situations the “price of anarchy” isn’t too high.
price of anarchy in the flow network game

• For flow $f$ let $\text{welfare}(f) := 1/(\text{average s-t-delay})$.

• In Braess’s paradox, the price of anarchy is $4/3$: by playing the NE the average delay is 2, but playing half-and-half on the upper and lower route, the average delay is $3/2$ (and that’s optimal).

• We have seen that the price of anarchy in network games can be arbitrarily high, when $x^d$ is an edge label.

• Remarkably, [Roughgarden-Tardos’00] showed that in a more general flow network setting (where there can be multiple source-destination pairs $(s_j, t_j)$), as long as “congestions” labeling edges are linear functions of $x$, the worst-case price of anarchy is $4/3$.

• In other words, for linear latencies, the Braess’s paradox example yields the worst-case scenario.
Back to atomic network congestion games

What is the price of anarchy in atomic network congestion games?

**Theorem:** [Christodoulou-Koutsoupias’2005]. *The price of anarchy for an NE in atomic network congestion games with linear utilities is*

\[
\frac{5}{2}
\]

*(And this is tight, just like 4/3 for non-atomic network congestion games.)*