Algorithmic Game Theory and Applications

Lecture 11: Games of Perfect Information

Kousha Etessami

finite games of perfect information

A perfect information (PI) game: 1 node per information set.

Theorem([Kuhn'53]) Every finite n-person extensive PI-game, $\mathcal{G}$, has a NE, in fact, a subgame-perfect NE (SPNE), in pure strategies.

I.e., some pure profile, $s^* = (s_1^*, \ldots, s_n^*)$, is a SPNE.

To prove this, we use some definitions. For a game $\mathcal{G}$ with game tree T, and for $w \in T$, define the **subtree** $T_w \subseteq T$, by: $T_w = \{w' \in T \mid w' = ww'' \text{ for } w'' \in \Sigma^*\}$.

Since tree is finite, we can just associate payoffs to the leaves. Thus, the subtee T_w, in an obvious way, defines a "**subgame**", $\mathcal{G}_w$, which is also a PI-game.

The **depth** of a node w in T is its length $|w|$ as a string. The depth of tree T is the maximum depth of any node in T. The depth of a game $\mathcal{G}$ is the depth of its game tree.

proof of Kuhn's theorem (backward induction)

Proof We prove by induction on the depth of a subgame $\mathcal{G}_w$ that it has a pure SPNE, $s^w = (s_1^w, \ldots, s_n^w)$. Then $s^* := s^\epsilon$.

Base case, depth 0: In this case we are at a leaf w. there is nothing to show: each player i gets payoff $u_i(w)$, and the strategies in the SPNE s^* are "empty" (it doesn't matter which player's node w is, since there are no actions to take.)

Inductive step: Suppose depth of $\mathcal{G}_w$ is $k+1$. Let $Act(w) = \{a'_1, \ldots, a'_r\}$ be the set of actions available at the root of $\mathcal{G}_w$. The subtrees $T_{wa'_j}$, for $j = 1, \ldots, r$, each define a PI-subgame $\mathcal{G}_{wa'_j}$, of depth $\leq k$.
Thus, by induction, each game $\mathcal{G}_{wa'_j}$ has a pure strategy SPNE, $s^{wa'_j} = (s_1^{wa'_j}, \ldots, s_n^{wa'_j})$.
To define $s^w = (s_1^w, \ldots, s_n^w)$, there are two cases to consider

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two cases

1. $w \in Pl_0$, i.e., the root node, w, of T_w is a chance node (belongs to "nature").
   Let the strategy s_i^w for player i be just the obvious "union" $\bigcup_{a' \in Act(w)} s_i^{wa'}$, of its pure strategies in each of the subgames.
   (Explanation of "union" of disjoint strategy functions.)
   Claim: $s^w = (s_1^w, \ldots, s_n^w)$ is a pure SPNE of $\mathcal{G}_w$. Suppose not. Then some player i could improve its expected payoff by switching to a different pure strategy in one of the subgames. But that violates the inductive hypothesis on that subgame.
2. $w \in Pl_i$, $i > 0$: the root, w, of T_w belongs to player i. For $a \in Act(w)$, let $h_i^{wa}(s^{wa})$ be the expected payoff to player i in the subgame $\mathcal{G}_{wa}$. Let $a' = \arg\max_{a \in Act(w)} h_i^{wa}(s^{wa})$. For players $i' \neq i$, define $s_{i'}^w = \bigcup_{a \in Act(w)} s_{i'}^{wa}$.
   For i, define $s_i^w = (\bigcup_{a \in Act(w)} s_i^{wa}) \cup \{w \mapsto a'\}$.
   Claim: $s^w = (s_1^w, \ldots, s_n^w)$ is a pure SPNE of $\mathcal{G}_w$. $\square$

algorithm for computing a SPNE in finite PI-games

The proof yields an EASY "bottom up" algorithm for computing a pure SPNE in a finite PI-game:

We inductively "attach" to the root of every subtree T_w, a SPNE s^w for the game $\mathcal{G}_w$, together with the expected payoff vector $h^w := (h_1^w(s^w), \ldots, h_n^w(s^w))$.

1. Initially: Attach to each leaf w the empty profile $s^w = (\emptyset, \ldots, \emptyset)$, & payoff vector $h^w := (u_1(w), \ldots, u_n(w))$.

2. **While** ($\exists$ unattached node w whose children are attached)
   - if $(w \in Pl_0)$ then
     $s^w := (s_1^w, \ldots, s_n^w)$, where $s_i^w := \bigcup_{a \in Act(w)} s_i^{wa}$;
     hence h^w is: $h_i^w(s^w) := \sum_{a \in Act(w)} q_w(a) * h_i^{wa}(s^{wa})$;

   else if $(w \in Pl_i \;\&\; i > 0)$ then
     Let $s^w := (s_1^w, \ldots, s_n^w)$, & $h^w := h^{wa'}$, where
     $a' := \arg\max_{a \in Act(w)} h_i^{wa}(s^{wa})$,
     $s_{i'}^w := \bigcup_{a \in Act(w)} s_{i'}^{wa}$, for $i' \neq i$, and
     $s_i^w := (\bigcup_{a \in Act(w)} s_i^{wa}) \bigcup \{w \mapsto a'\}$;

consequences for zero-sum finite PI-games

Recall that, by the Minimax Theorem, for every finite zero-sum game Γ, there is a value v^* such that for any NE (x_1^*, x_2^*) of Γ, $v^* = U(x_1^*, x_2^*)$, and

$$\max_{x_1 \in X_1} \min_{x_2 \in X_2} U(x_1, x_2) = v^* = \min_{x_2 \in X_2} \max_{x_1 \in X_1} U(x_1, x_2)$$

But it follows from Kuhn's theorem that for extensive PI-games $\mathcal{G}$ there is in fact a pure NE (in fact, SPNE) (s_1^*, s_2^*) such that $v^* = u(s_1^*, s_2^*) := h(s_1^*, s_2^*)$, and thus that in fact

$$\max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2) = v^* = \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2)$$

Definition A finite zero-sum game Γ is **determined**, if

$$\max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2) = \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2)$$

It thus follows from Kuhn's theorem that:

Proposition ([Zermelo'1912]) Every finite zero-sum PI-game, $\mathcal{G}$, is determined. Moreover, the value & a pure minimax profile can be computed "efficiently" from $\mathcal{G}$.

chess

Chess is a finite PI-game (after 50 moves with no piece taken, it ends in a draw). In fact, it's a **win-lose-draw** PI-game: no chance nodes possible payoffs are 1, -1, and 0.

Proposition([Zermelo'1912]) In Chess, either:

1. White has a "winning strategy", or
2. Black has a "winning strategy", or
3. Both players have strategies to force a draw.

A "**winning strategy**", e.g., for White (Player 1) is a pure strategy s_1^* that guarantees value $u(s_1^*, s_2) = 1$, for all s_2.

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Question: Which one is the right answer??

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Problem: The tree is far too big!!

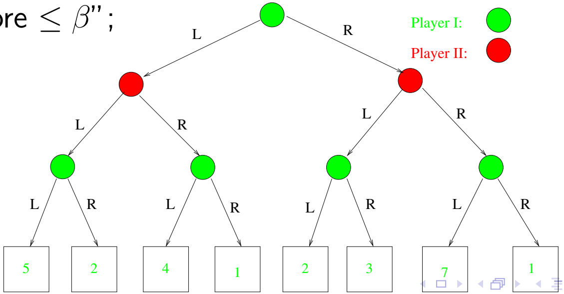
Even with $\sim$ 200 depth & $\sim$ 5 moves per node:

$$5^{200} \text{ nodes!}$$

Despite having an "efficient" algorithm to compute the value v^* given the tree, we can't even look at the whole tree! We need algorithms that don't look at the whole tree.

50 years of game-tree search

There's $>$ 50 years of research on chess & other game playing programs, (Shannon, Turing, ...). Heuristic game-tree search is now very refined. See any AI text (e.g., [Russel-Norvig]).
If we have a function *Eval*(w) that heuristically "evaluates" a node's "goodness" score, we can use *Eval*(w) to stop the search at, e.g., desired depth. While searching "top-down", we can "prune out" irrelevant subtrees using α-β-**pruning**. Idea: while searching minmax tree, maintain two values: α- "maximizer can assure score $\geq \alpha$"; & β- "minimizer can assure score $\leq \beta$";



minmax search with α-β-pruning

Assume, for simplicity, that players alternate moves, root belongs to Player 1 (maximizer), and $-1 \leq \mathit{Eval}(w) \leq +1$. Score -1 $(+1)$ means player 1 definitely loses (wins). Start the search by calling: $\mathbf{MaxVal}(\epsilon, -1, +1)$;

$\mathbf{MaxVal}(w, \alpha, \beta)$

If $\mathit{depth}(w) \geq \mathit{MaxDepth}$ then **return** $\mathit{Eval}(w)$.

Else, for each $a \in \mathit{Act}(w)$

$\alpha := \max\{\alpha, \mathbf{MinVal}(wa, \alpha, \beta)\}$;

if $\alpha \geq \beta$, then **return** β

return α

$\mathbf{MinVal}(w, \alpha, \beta)$

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Else, for each $a \in \mathit{Act}(w)$

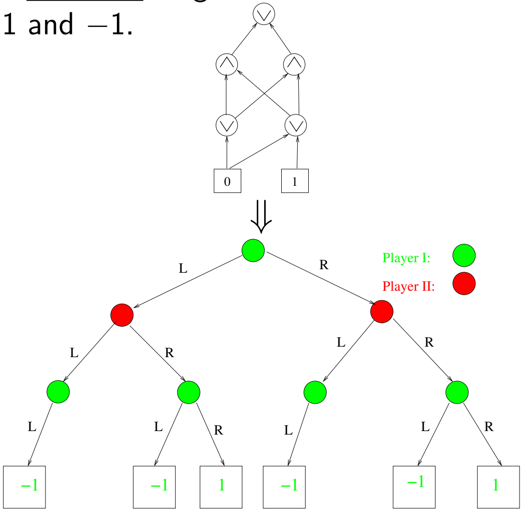
$\beta := \min\{\beta, \mathbf{MaxVal}(wa, \alpha, \beta)\}$;

if $\beta \leq \alpha$, then **return** α

return β

boolean circuits as finite PI-games

Boolean circuits can be viewed as a zero-sum PI-game, between AND and OR: OR the maximizer, AND the minimizer: a **<u>win-lose</u>** PI-game: no chance nodes & only payoffs are 1 and -1.



Let's generalize to infinite zero-sum PI-games

For a (possibly infinite) zero-sum 2-player PI-game, we would like to similarly define the game to be "determined" if

$$\max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2) = \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2)$$

But, for infinite games max & min may not exist! Instead, we call an (infinite) zero-sum game **determined** if:

$$\sup_{s_1 \in S_1} \inf_{s_2 \in S_2} u(s_1, s_2) = \inf_{s_2 \in S_2} \sup_{s_1 \in S_1} u(s_1, s_2)$$

In the simple setting of infinite win-lose PI-games (2 players, zero-sum, no chance nodes, and only payoffs are 1 and -1), this definition says a game is determined precisely when one player or the other has a **winning strategy**: a strategy $s_1^* \in S_1$ such that for any $s_2 \in S_2$, $u(s_1^*, s_2) = 1$ (and vice versa for player 2).

Question: Is every win-lose PI-game determined?

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Question: Is every win-lose PI-game determined?

Answer: No ....

determinacy and its boundaries

For win-lose PI-games, we can define the payoff function by providing the set $Y = u_1^{-1}(1) \subseteq \Psi_T$, of complete plays on which player 1 wins (player 2 necessarily wins on all other plays).

If, additionally, we assume that players alternate moves, we can specify such a game as $\mathcal{G} = \langle T, Y \rangle$.

Fact For tree $T = \{L, R\}^*$, there are sets $Y \subseteq \Psi_T$, such that the win-lose PI-game $\mathcal{G} = \langle T, Y \rangle$ is **not** determined.

(Proof uses the "axiom of choice". See, e.g., [Mycielski, Ch. 3 of Handbook of GT,1992].)

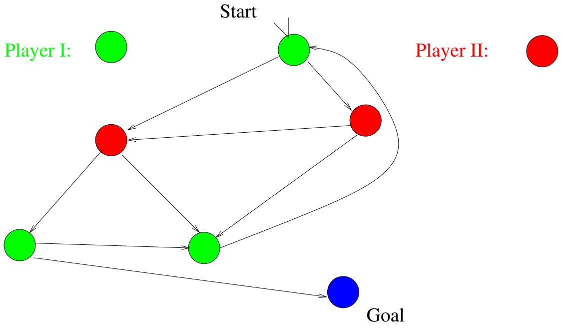
Fortunately, large classes of win-lose PI-games are determined:

Theorem([D. A. Martin'75]) Whenever Y is a so called "*Borel set*", the game $\langle \Sigma^*, Y \rangle$ is determined.

(A deep theorem, with connections to logic and set theory. Theorem holds even when the action alphabet Σ is infinite.)

food for thought: win-lose games on finite graphs

Instead of a tree, we have a finite directed graph:



▷ Starting at "Start", does Player I have a strategy to "force" the play to reach the "Goal"?

▷ Note: this is a (possibly infinite) win-lose PI-game.

▷ Is this game determined for all finite graphs?

▷ If so, how would you compute a winning strategy for Player 1?