Algorithmic Game Theory and Applications:
Coursework 2

Due ONLINE (submit all solutions as one ** PDF ** file, via LEARN) at 3:00pm on Friday, *** April 3rd *** (note the extended deadline). (This is a firm deadline.) Please write your ID number on the coursework that you hand in. Please do not collaborate with other students on your coursework. Your solutions must be your own.

1. Recall that a Nash equilibrium in an extensive game is subgame perfect nash equilibrium (SPNE) if it is also a Nash equilibrium in every subgame of the original game. Formally, a subgame, is a game defined by a subtree, $T_v$ of the original game tree, $T$, such that the subtree $T_v$, rooted at a node $v$, has the property that for every descendent $u$ of $v$ in the game tree (including $v$ itself), every node in the same information set as $u$ is also in the subtree $T_v$.

(a) [7 points] Give an example of a pure NE which is not a SPNE, for a finite extensive form game of perfect information.

(b) [10 points] Prove that every finite extensive game of perfect information where there are no chance nodes and where no player gets the same payoff at any two distinct leaves, must have a unique pure-strategy SPNE.

(c) [8 points] Give an example of a finite extensive form game that contains a pure Nash Equilibrium but does not contain any subgame-perfect pure Nash Equilibrium. Justify your answer.

2. Consider the finite extensive form game of imperfect information depicted in Figure 1.

(a) [3 points] Does this game satisfy “perfect recall”? Explain your answer.

(b) [12 points] Compute all SPNEs in this game, describing each of them in terms of “behavior strategies”.

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Figure 1: Question 2
(c) [6 points] Are there any other Nash equilibria other than SPNEs you have computed? Explain your answer.

(d) [4 points] Which of the equilibria in this game (pure and mixed) are “credible” and “sequentially rational” (specifically, which ones do not involve “non-credible threats”). Explain.

3. Consider the atomic network congestion game, with three players, described by the directed graph in Figure 2.

In this game, every player $i$ (for $i = 1, 2, 3$) needs to choose a directed path from the source $s$ to the target $t$. Thus, every player $i$’s set of possible actions (i.e., its set of pure strategies) is the set of all possible directed paths from $s$ to $t$.

Each edge $e$ is labeled with a sequence of three numbers $(c_1, c_2, c_3)$. Given a profile $\pi = (\pi_1, \pi_2, \pi_3)$ of pure strategies (i.e., $s$-$t$-paths) for all three players, the cost to player $i$ of each directed edge, $e$, that is contained in player $i$’s path $\pi_i$, is $c_k$, where $k$ is the total number of players that have chosen edge $e$ in their path. The total cost to player $i$, in the given profile $\pi$, is the sum of the costs of all the edges in its path $\pi_i$ from $s$ to $t$. Each player of course wants to minimize its own total cost.

(a) [10 points] Compute a pure strategy NE in this atomic network congestion game. Explain why what you have computed is a pure NE.

(b) [5 points] Is the pure NE you have computed unique? Explain.
(c) [10 points] Let the social welfare of a profile of strategies in an atomic network congestion game be defined as the inverse of the sum total cost for all players in that profile. Compute the pure price of anarchy in this atomic network congestion game, where the pure price of anarchy here is defined as the ratio of the best (i.e., maximum) social welfare of any pure profile, divided by the worst (i.e., minimum) social welfare of any pure NE.

4. The auction house Christie’s of London is auctioning a triptych (a series of three related painting) by the famous artist Francis Bacon, entitled “Three Studies of Isabel Rawsthorne”. We will refer to the three paintings in the triptych series as T1, T2, and T3, respectively.

Suppose that Christie’s hires you as an auction designer, and suppose that you decide to use the Vickery-Clarke-Groves mechanism as an auction, in order to determine which bidder should get which part(s) of the triptych, and at what price. Suppose that there are only three bidders. The three bidders’ names are: Susanne (S), Lakshmi (L), and Bill (B).

Since you are running a VCG-based auction, you ask each bidder to give you their valuation for every subset of the paintings in the triptych, as part of the bidding process. Suppose that the valuation functions \( v_S \), \( v_L \), and \( v_B \) that you receive from the three bidders, S, L, and B, respectively, are as follows (the numbers denote 10^5 pounds):

<table>
<thead>
<tr>
<th>bidder i</th>
<th>( v_i(\emptyset) )</th>
<th>( v_i(T_1) )</th>
<th>( v_i(T_2) )</th>
<th>( v_i(T_3) )</th>
<th>( v_i(T_1,T_2) )</th>
<th>( v_i(T_1,T_3) )</th>
<th>( v_i(T_2,T_3) )</th>
<th>( v_i(T_1,T_2,T_3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>i := S</td>
<td>0</td>
<td>17</td>
<td>11</td>
<td>19</td>
<td>31</td>
<td>42</td>
<td>33</td>
<td>60</td>
</tr>
<tr>
<td>i := L</td>
<td>0</td>
<td>13</td>
<td>21</td>
<td>17</td>
<td>38</td>
<td>30</td>
<td>43</td>
<td>61</td>
</tr>
<tr>
<td>i := B</td>
<td>0</td>
<td>20</td>
<td>14</td>
<td>12</td>
<td>39</td>
<td>36</td>
<td>40</td>
<td>62</td>
</tr>
</tbody>
</table>

(a) [16 points] Give a VCG outcome for this auction. In other words, specify, in that VCG outcome, which bidders will get which of the painting(s), and what price they will each pay for the painting(s) they get. Justify your answer, and show your calculations.

(b) [6 points] Is the VCG outcome you have calculated in part (a) unique? Justify your answer, and show your calculations.
(c) [3 points] Comment on the wisdom of choosing the VCG mechanism for this auction. Do you think it is a good idea to do so? What if instead of this triptych, Christie’s wanted to do a simultaneous auction of 20 Andy Warhol paintings, and they knew that at least 30 viable bidders want to bid for (subsets of) those paintings. Would you suggest using the VCG mechanism for such an auction? What alternative auction would you use, and why? Explain, briefly.