Algorithmic Game Theory and Applications: Homework 1

October 5, 2009

This homework is due at 3:00pm, Wednesday, October 28th.
(This is a firm deadline. Please hand it in by that time to the ITO).

1. Consider the following 2-player strategic game:

$$\begin{bmatrix} (4,9) & (6,2) \\ (7,3) & (3,8) \end{bmatrix}$$

   This is a "bimatrix", to be read as follows: Player 1 is the row player, and Player 2 is the column player. If the content of the bimatrix at row i and column j is the pair (a,b), then $u_1(i,j) = a$ and $u_2(i,j) = b$.

   (a) Consider the mixed strategies $x_1 = (1/4, 3/4)$ and $x_2 = (2/3, 1/3)$, for player 1 and 2, respectively. Here, e.g., player 1 is playing row 2 with probability 3/4, etc.

   What is the expected payoff to each player under this profile?

   (b) Does this game have a pure Nash equilibrium? If so, point one out, if not, show why it does not, and in any case, find a Nash equilibrium for this game.

   (c) We discussed this is class, but didn't write it formally: explain why if x is a Nash Equilibrium then it is a fixed point of the function $f : X \mapsto X$ defined during the proof of Nash's Theorem.

2. In this simple exercise you are asked to prove that the Fourier-Motzkin elimination algorithm for LP, presented in lecture 5, is correct. Specifically, prove the following. Suppose that in the for loop of the algorithm, prior to some iteration i, we have a set of linear constraints,

$C(x_0, x_1, \ldots, x_i)$, in the variables $x_0, \ldots, x_i$ and after the elimination of (contraints involving) variable x_i we have a new set of constraints $C'(x_0, \ldots, x_{i-1})$ in the remaining variables. Prove that, if there is any solution $x_0^*, \ldots, x_{i-1}^*$ satisfying the constraints C', then this solution can be extended, with a value x_i^*, to a solution $x_0^*, \ldots, x_i^*$ of the contraints C. Then use this claim to show that the algorithm solves the LP problem correctly in all cases.

3. Recall the definition of a (finite) 2-player symmetric game.

   (a) Show that every finite symmetric 2-player zero-sum game has minimax value 0. You can not rely on the proof in the "symmetries of games" section of Nash's paper for this, and must show this just assuming the minimax theorem.
   Show, moreover, that the minmaximizing strategies for player 1 are exactly the maxminimizing strategies for player 2.

   (b) Construct a 2-player zero-sum game, where if your opponent (player 2) plays "completely randomly" (i.e., every strategy has equal probability), then you (player 1) are better off not playing a minmaximizer.

4. Consider the 2-player zero-sum game given by:

$$\begin{bmatrix} 4 & 2 & 9 & 2 & 5 \\ 6 & 1 & 5 & 9 & 2 \\ 1 & 4 & 8 & 1 & 7 \\ 5 & 1 & 3 & 5 & 6 \end{bmatrix}$$

   Use the simplex implementation in xmaple, available on DICE machines, to compute the minimax value of this game, and also to find "optimal" (i.e., minmaximizer, etc.) strategies for each player.

   Specify the LPs you used to solve the game.

   (To run xmaple, type "xmaple" or "smaple12" at the shell command prompt. To use simplex on xmaple, type "? simplex" at the command prompt within xmaple. This will give you adequate information and examples on how to specify LPs and solve them in xmaple.)

5. (a) Consider the following "revised LP problem": the LP is given by constraint set $C(x_1, \ldots, x_n)$, together with an ordered listing of linear objective functions together with optimization criteria

for each as follows: $((f_1, Opt_1), \ldots, (f_r, Opt_r))$. Here each f_j is a linear objective and each Opt_j is either `Maximize` or `Minimize`.

The interpretation of this "revised LP problem" is as follows: you wish to find a solution vector $v \in K(C)$, such that: v optimizes f_1 according to criterion Opt_1, and, *among those vectors that do so*, v optimizes f_2 according to criterion Opt_2, and *among those vectors that do so*, ..., and so on.

(I hope this is clear. If not, ask.)

Give an efficient algorithm for solving this revised LP problem.

(b) Find the dual to the following LP:

Maximize $x_1 + 3x_2 + 2x_3$
Subject to:
$4x_1 + x_2 \leq 9$
$3x_1 + 2x_3 \leq 4$
$5x_2 + 3x_3 \leq 10$
$x_1, x_2, x_3 \geq 0$

Then, solve the LP and its dual, using xmaple if you wish.

6. (This is a somewhat more challenging problem than the prior problems. But still very doable.)

One variant of the Farkas Lemma says the following:

Farkas Lemma A linear system of inequalities $Ax \leq b$ has a solution x if and only if there is no vector y satisfying $y \geq 0$ and $y^T A = 0$ (i.e., 0 in every coordinate) and such that $y^T b < 0$.

Prove this Farkas Lemma with the aid of Fourier-Motzkin elimination. (*Hint:* use induction on the number of columns of A.)