Algorithmic Game Theory and Applications: Homework 1

This homework is due at 3:00pm, on Thursday, February 25th.
This is a firm deadline. Please submit your solutions online as PDF files, using the LEARN page for AGTA. (Instructions for how to submit the PDF files on LEARN will be provided separately to all students via email and via the course web page.) *Do not collaborate with other students on the coursework. Your solutions must be your own.*
Each question counts for 25 points, for a total of 100 points.

1. Consider the following 2-player strategic game, G:

$$\begin{bmatrix} (1,3) & (7,5) & (5,4) & (9,8) \\ (0,7) & (6,8) & (5,9) & (8,8) \\ (1,1) & (4,0) & (2,1) & (3,1) \\ (1,2) & (8,6) & (6,7) & (7,6) \end{bmatrix}$$

   This is a "bimatrix", to be read as follows: Player 1 is the row player, and Player 2 is the column player. If the content of the bimatrix at row i and column j is the pair (a, b), then $u_1(i, j) = a$ and $u_2(i, j) = b$.

   (a) (18 points)
   Compute *all* of the Nash equilibria (NEs) of this game G, together with the expected payoff to each player in each NE.
   Explain why any profile x that you claim is an NE of G, is indeed an NE of G.
   Furthermore, explain why there are no other (pure or mixed) NEs of G, other than the profile(s) you claim are NE(s) of G.

   (b) (7 points) For a game H, let NE(H) denote the set of all (pure or mixed) NE's of the game H. For a mixed strategy $x_1 \in X_1$ for player 1, define:

$$g_1(x_1) := \begin{cases} 2 & \text{if } x_{1,1} \geq 1/2 \\ 1 & \text{otherwise} \end{cases} \tag{1}$$

   Recall $\pi_{i,j}$ denotes the j'th pure strategy of player i. Show that there exist a 3-player finite normal form game, G', with pure strategy sets $S_1 = S_2 = \{1, 2, 3, 4\}$, and $S_3 = \{1, 2\}$ for the three players, such that:

$$\text{NE}(G') = \{(x_1, x_2, \pi_{3,g_1(x_1)}) \mid (x_1, x_2) \in \text{NE}(G)\}.$$

2. (a) (10 points) Consider the 2-player zero-sum game given by the following payoff matrix for player 1 (the row player):

$$\begin{bmatrix} 1 & 2 & 7 & 2 & 4 \\ 0 & 0 & 9 & 6 & 2 \\ 7 & 9 & 4 & 5 & 3 \\ 1 & 4 & 0 & 7 & 9 \\ 9 & 7 & 3 & 8 & 3 \end{bmatrix}$$

Compute both the minimax value for this game, as well as a minimax profile (NE), i.e., "optimal" (i.e., minmaximizer and maxminimizer) strategies for both players 1 and 2, respectively.

(You can, for example, use the linear programming solver package `linprog` in MATLAB, available on DICE machines, to do this. To run MATLAB, type "matlab" at the shell command prompt. For guidance on using the `linprog` package, see: `http://uk.mathworks.com/help/optim/ug/linprog.html`.)

(b) (15 points) Recall from Lecture 7 on LP duality, the *symmetric* 2-player zero-sum game, G, for which the (skew-symmetric) payoff matrix (in block form) for player 1 is:

$$B = \begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix}$$

Suppose that there exist vectors $x' \in \mathbb{R}^n$ and $y' \in \mathbb{R}^m$, such that $Ax' < b$, $x' \geq 0$, $A^T y' > c$ and $y' \geq 0$. (Note the two *strict* inequalities.) Prove that for the game G, every minmaximizer strategy $w = (y^*, x^*, z)$ for player 1 (and hence also every maxminimizer strategy for player 2, since the game is symmetric) has the property that $z > 0$, i.e., the last pure strategy is played with positive probability. (Recall that this was one of the missing steps in our sketch proof in the lecture that the minimax theorem implies the LP duality theorem.)

(Hint: Let $w = (y^*, x^*, z)$ be a maxminimizer strategy for player 2 in the game G. Note that the value of any symmetric 2-player zero-sum game must be equal to zero. This implies, by the minimax theorem, that $Bw \leq 0$. Suppose, for contradiction, that $z = 0$. What does this imply about Ax^* , $A^T y^*$, and $b^T y^* - c^T x^*$? Then if $y^* \neq 0$, show that this implies $(y^*)^T(Ax' - b) < 0$. In turn, show that it also implies $(x^*)^T(A^T y' - c) > 0$. Use these and related facts to conclude a contradiction.)

3. Consider either of the following simple *2-player zero-sum* games, where the payoff table for Player 1 (the row player) is given by:

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{or} \quad B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

We can view both of these games as a game where each player chooses "heads" (H) or "tails" (T), where the first strategy for each player is denoted H and the second strategy is denoted T.

Choose ONE of the above two games, either A or B.

(a) (5 points) First, what is the unique Nash equilibrium, or equivalently the unique minimax profile of mixed strategies for the two players, in the game you have chosen? And what is the minimax value of that game?

(b) (20 points) Now, suppose that the two players play the same game you have chosen in part (a), against each other, over and over again, for ever, and suppose that both of them use the following method in order to update their own strategy after each round of the game.

i. At the beginning, in the first round, each player chooses either of the pure strategies, H or T, arbitrarily, and plays that.

ii. After each round, each player i accumulates statistics on how its opponent has played until now, meaning how many Heads and how many Tails have been played by the opponent, over all rounds of the game played thusfar. Suppose these numbers are N Heads and M Tails.

Then player i uses these statistics to "guess" its opponents *"statistical mixed strategy"* as follows. It assumes that its opponent will next play a mixed strategy σ, which plays Heads with probability $N/(N+M)$ and plays Tails with probability $M/(N+M)$.

Under the assumption that its opponent is playing the *"statistical mixed strategy"* σ, in the next round player i plays a pure strategy (H or T) that is a pure *best response* to σ.

If both H and T are a best response at any round, then player i can use **any tie breaking rule it wish** in order to determine the pure strategy it plays in the next round.

iii. They repeat playing like this forever.

Prove that, regardless how the two players start playing the game in the first round, the "statistical mixed strategies" of both players in this method of repeatedly playing the game will, in the long run, as the number of rounds goes to infinity, converge to their mixed strategies in the unique Nash equilibrium of the game.

You are allowed to show that this holds using any specific tie breaking rule that you want. Please specify the precise tie breaking rule you have used. (It turns out that it hold true for any tie breaking rule. But some tie breaking rules may make the proof easier than others.)

Alternatively, instead of proving that it works, you can "show" this **experimentally** by writing a simple program that plays this strategy update method for both players in repeated play, and show experimentally that, for all possible start strategies of both players, the "statistical mixed strategies" of the two players *looks like* it is converging to their NE strategies. (You will need to provide a graph of your experimental output which shows that convergence "looks like" it is happening.)

(Note that experimentally you can only "show" that the "statistical mixed strategies" *look like* they are converging to the NE, by repeating the game some finite number of times, but you can not be sure that they do actually converge to the NE, without proving this.)

4. (a) (20 points) One variant of the Farkas Lemma says the following:

   Farkas Lemma A linear system of inequalities $Ax \leq b$ has a solution x if and only if there is no vector y satisfying $y \geq 0$ and $y^T A = 0$ (i.e., 0 in every coordinate) and such that $y^T b < 0$.

   Prove this Farkas Lemma with the aid of Fourier-Motzkin elimination. (*Hint:* One direction of the "if and only if" is easy. For the other direction, use induction on the number of columns of A, using the fact that Fourier-Motzkin elimination "works". Note basically that each round of Fourier-Motzkin elimination can "eliminate one variable" by pre-multiplying a given system of linear inequalities by a *non-negative* matrix.)

   (b) (5 points) Recall that in the *Strong Duality Theorem* one possible case (case 4, in the theorem as stated on our lecture slides) is that *both* the primal LP and its dual LP are *infeasible*. Give an example of a primal LP and its dual LP, for which both are infeasible (and argue why they are both infeasible).