Algorithms and Data Structures 2020/21
Notes for week 4

1. The recurrence (as usual n is $j - i + 1$) is

$$T_{RM}(n) = \begin{cases} 1 & \text{if } n = 1 \\ T_{RM}(\lfloor \frac{n}{2} \rfloor) + T_{RM}(\lceil \frac{n}{2} \rceil) + 4 & \text{if } n > 1 \end{cases}$$

   The 1 for the case of $n = 1$ comes from the observation that the only work done in this case is the comparison of i and j. The 4 in the recursive case comes from the $i < j$ test, the assignment to m (I guess it is debatable how many operations this corresponds to), the test $\ell < r$, and the subsequent **return**.

   Then using the Master theorem, we have $k = 0$ and $c = 1$. Hence running time is $\Theta(n)$.

2. (a) We prove $T(\widehat{n}) = (\widehat{n})^2(1 + \lg(\widehat{n}))$ for all powers-of-2 by induction.

   base case: $p = 0$ and $n = 1$. Then $T(1) = 1$ by definition. Also $1^2(1 + \lg(1)) = 1^2(1 + 0) = 1$. So true.

   Induction hypothesis (IH): $T(\widehat{n}) = (\widehat{n})^2(1 + \lg(\widehat{n}))$ for $\widehat{n} = 1, \ldots 2^p$.

   Induction step: We must prove that under the (IH), that the claim also holds for $\widehat{n} = 2^{p+1}$.

   We have $p + 1 \geq 1$, so $2^{p+1} \geq 2$, so we can apply the recurrence to get

$$\begin{aligned} T(2^{p+1}) &= 4T(\lfloor 2^{p+1}/2 \rfloor) + 2^{2(p+1)} \\ &= 4T(2^p) + 2^{2(p+1)} && \text{(because } 2^{p+1}/2 = 2^p \in \mathbb{N}) \\ &= 4(2^p)^2(1 + \lg(2^p)) + 2^{2(p+1)} && \text{(by (IH))} \\ &= 2^{2p+2}(1 + \lg(2^p)) + 2^{2(p+1)} \\ &= 2^{2p+2}(\lg(2^{p+1})) + 2^{2(p+1)} && \text{(by } \lg(2 \cdot 2^p) = \lg(2) + \lg(2^p) = 1 + \lg(2^p)) \\ &= 2^{2p+2}(1 + \lg(2^{p+1})), \end{aligned}$$

   as required.

   Note that the first line is obtained by substituting $\widehat{n} = 2^{p+1}$ into the recurrence; the second line is by observing that $\lfloor \cdot \rfloor$ is unnecessary as $2^{p+1}/2 = 2^p$ is an integer; the third line is due to substituting the (IH) for $T(2^p)$, 2^p being *strictly* smaller than 2^{p+1}; the fourth and fifth lines come from applying multiplication and properties-of-logs directly; and the final line by rearranging terms.

(b) We can just prove $T(n) \leq T(n+1)$ for all $n \in \mathbb{N}$. Then we can use *transitivity* to observe that $T(j) \leq T(k)$ for all $j < k, j, k \in \mathbb{N}$. *there are other ways, eg working explicitly with n and m, but the (IH) would be slightly messier in wording - eg, see slide 14 of lectures 2-3: where the (IH) is less tidy.*

First we prove the base case.

Base case: $k = 1$. We have $T(1) = 1$; however $T(2) = 4 \cdot T(1) + 2^2 = 8$; clearly $T(1) < T(2)$.

Next we formulate our Induction Hypothesis.

Induction Hypothesis (IH): *for every* $k, 1 \leq k < n$, we have $T(k) < T(k+1)$.

Induction step: Based on the (IH) for all $k < n$, we will show $T(n) \leq T(n+1)$ also. Note we must have $n \geq 2$ (else we'd be in the base case), so the recursive step of the recurrence applies to both $T(n)$ and also $T(n+1)$. We can write

$$\begin{aligned} T(n) &= 4T(\lfloor \frac{n}{2} \rfloor) + n^2 \\ T(n+1) &= 4T(\lfloor \frac{n+1}{2} \rfloor) + (n+1)^2 \end{aligned}$$

Now observe that *either*

$$\lfloor \tfrac{n+1}{2} \rfloor = \lfloor \tfrac{n}{2} \rfloor \text{ or } \lfloor \tfrac{n+1}{2} \rfloor = \lfloor \tfrac{n}{2} \rfloor + 1.$$

In the first case (n even, $\lfloor \frac{n+1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$), we have $4T(\lfloor \frac{n+1}{2} \rfloor) = 4T(\lfloor \frac{n}{2} \rfloor)$.

In the second case (n odd) the (IH) can be applied to $\lfloor \frac{n}{2} \rfloor$ because $\lfloor \frac{n}{2} \rfloor \leq n$ (this is true always when $n \geq 2$). Hence the (IH) tells us that $4T(\lfloor \frac{n}{2} \rfloor) < 4T(\lfloor \frac{n+1}{2} \rfloor)$.

We get $4T(\lfloor \frac{n}{2} \rfloor) \leq 4T(\lfloor \frac{n+1}{2} \rfloor)$ in either case.

Also $n^2 < (n+1)^2$. Combining these two facts, we get that overall $T(n) < T(n+1)$ (ie, given the (IH), the claim holds for n also)

By induction, we have $T(n) < T(n+1)$ for all $n \in \mathbb{N}$.

(*) Note we really *needed* a recurrence with $=$ and with explicit constants (no O, no Θ) to prove the strictly increasing. This is because we substituted the T on the right-hand side *and* the left-hand side of the claim $T(j) < T(k)$.

(c) Now consider an arbitrary $n \in \mathbb{N}$. Let p be the greatest integer such that $2^p \leq n$ (note we are then guaranteed $2^p > n/2$).

By (a), $T(2^p) = (2^p)^2(1 + \lg(2^p))$. By (b), we know that $T(n) \geq T(2^p)$.

By above $2^p > n/2$. Hence we have

$$\begin{aligned} T(n) \geq T(2^p) &= (2^p)^2(1 + \lg(2^p)) \\ &> (n/2)^2(1 + \lg(n/2)) \\ &= (n^2/4)(\lg(n)). \end{aligned}$$

This gives $\Omega(n^2 \lg(n))$ for $n_0 = 1$ and $c = 1/4$.

3. Use Strassen's algorithm to compute the matrix product

$$\begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} 8 & 4 \\ 6 & 2 \end{pmatrix}.$$

Just set up the P1 -P7 equations on the board, multiply them out, then evaluate $C_{11}, C_{12}, C_{21}, C_{22}$. You'll need to have lecture 4 (or the book) along with you.

4. Describe an algorithm for efficiently multiplying a $(p \times q)$ matrix with a $(q \times r)$ matrix, where p, q, r are arbitrary positive integers. The running time should be $\Theta(n^{\lg(7)})$, where $n = \max\{p, q, r\}$.

Answer:
Let A be the $p \times q$ matrix, and B be the $q \times r$ matrix. We round up the matrices to become $n \times n$ matrices A', B', keeping A in the top lhs of A' (and similarly B in the top lhs of B'). All the entries outside the top-left $p \times q$ of A' are 0 and similarly for entries outside the top-left $q \times r$ of B'.

We call $\text{Strassen}(A', B')$ and then extract the top-left hand $p \times r$ matrix.

For this algorithm it's clear that the runtime is $\Theta(n^{\lg(7)})$ (because that is the running time of Strassen on $n \times n$ matrices, and because the "extra work" in mapping to-and-from $n \times n$ matrices is only $O(n^2)$).

Observation: A tangential issue wrt this algorithm is that for this general "rectangular" case it is NOT clear that this "reduce to Strassen" algorithm is often a good strategy. Suppose wlog that $p = \max\{p, q, r\}$. Then the naïve matrix multiplication algorithm is $\Theta(pqr)$. Our asymptotic running-time from "reduce to Strassen" is only better if $qr \geq p^{\lg(7)-1} \sim p^{1.8}$, which is not necessarily the case in the "rectangular" setting.