The Master Theorem for solving recurrences

Theorem 3.1

Let $n_0 \in \mathbb{N}$, $k \in \mathbb{N}_0$ *and* $a, b \in \mathbb{R}$ *with* $a > 0$ *and* $b > 1$, *and let* $T : \mathbb{N} \to \mathbb{R}$ *satisfy the following recurrence:*

$$T(n) = \begin{cases} \Theta(1) & \text{if } n < n_0, \\ a \cdot T(n/b) + \Theta(n^k) & \text{if } n \geq n_0. \end{cases}$$

Let $c = \log_b(a)$; *we call* c *the* ***critical exponent***. *Then*

$$T(n) = \begin{cases} \Theta(n^c) & \text{if } k < c & \text{(I)}, \\ \Theta(n^c \cdot \lg(n)) & \text{if } k = c & \text{(II)}, \\ \Theta(n^k) & \text{if } k > c & \text{(III)}. \end{cases}$$

The n/b *in the recurrence may stand for both* $\lfloor n/b \rfloor$ *and* $\lceil n/b \rceil$. *More precisely, the theorem holds if we replace* $a \cdot T(n/b)$ *in the recurrence by* $a_1 \cdot T(\lfloor n/b \rfloor) + a_2 \cdot T(\lceil n/b \rceil)$ *for any* $a_1, a_2 \geq 0$ *with* $a_1 + a_2 = a$.

The Master Theorem (cont'd)

- We don't have time to prove the Master Theorem in class. You can find the proof in Section 4.4 of [CLRS]. *Section 4.4 of [CLR]*. Their version of the M.T. is a bit more general than ours.
- **Homework:** To get a feel for the Master Theorem, consider the following examples:

$$\begin{aligned} T(n) &= 4T(n/2) + n, \\ T(n) &= 4T(\lfloor n/2 \rfloor) + n^2, \\ T(n) &= 4T(n/2) + n^3. \end{aligned}$$

  Use unfold-and-sum to answer the first and third of these. We solved the second one *by first principles* in lecture 2, and it hurt! (mostly because of the $\lfloor \ \rfloor$).

Matrix Multiplication

Recall

The product of two $(n \times n)$-matrices

$$A = (a_{ij})_{1 \leq i,j \leq n} \quad \text{and} \quad B = (b_{ij})_{1 \leq i,j \leq n}$$

is the $(n \times n)$-matrix $C = AB$ where $C = (c_{ij})_{1 \leq i,j \leq n}$ with entries

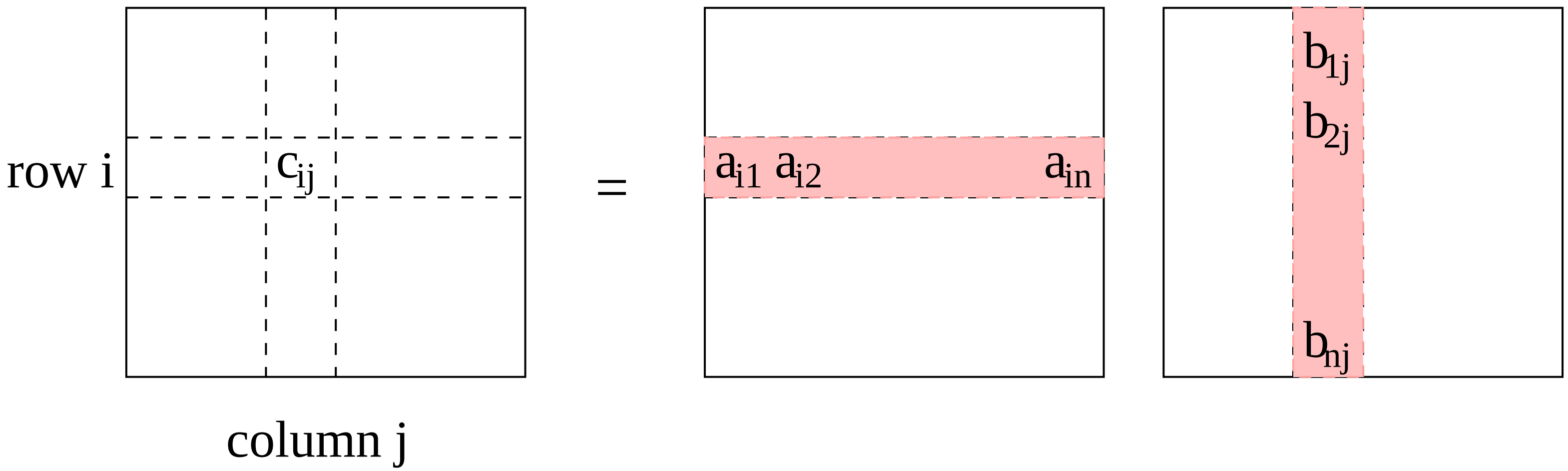
$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

The Matrix Multiplication Problem

Input: $(n \times n)$-matrices A and B

Output: the $(n \times n)$-matrix AB

Matrix Multiplication



- n multiplications and n additions for each c_{ij}.
- there are n^2 different c_{ij} entries.

A straightforward algorithm

Algorithm $\textsc{MatMult}(A, B)$

1. $n \leftarrow$ number of rows of A
2. **for** $i \leftarrow 1$ **to** n **do**
3.   **for** $j \leftarrow 1$ **to** n **do**
4. $c_{ij} \leftarrow 0$
5.   **for** $k \leftarrow 1$ **to** n **do**
6. $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$
7. **return** $C = (c_{ij})_{1 \leq i,j \leq n}$

Requires

$$\Theta(n^3)$$

arithmetic operations (additions and multiplications).

A naive divide-and-conquer algorithm

Observe

If

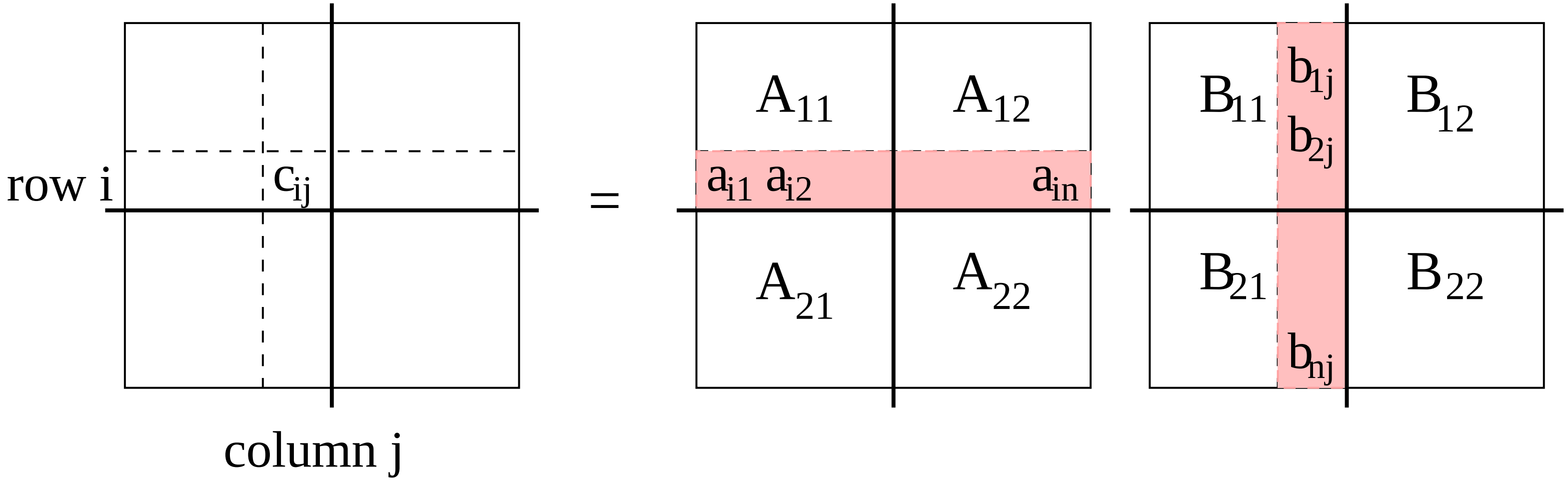
$$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \quad \text{and} \quad B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$$

for $(n/2 \times n/2)$-submatrices A_{ij} and B_{ij} then

$$AB = \left(\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right)$$

note: We are assuming n is even.

A naive divide-and-conquer algorithm



Suppose $i \le n/2$ and $j \le n/2$. Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \underbrace{\sum_{k=1}^{n/2} a_{ik} b_{kj}}_{\in A_{11}B_{11}} + \underbrace{\sum_{k=n/2+1}^{n} a_{ik} b_{kj}}_{\in A_{12}B_{21}}$$

A naive divide-and-conquer algorithm (cont'd)

Assume n is a power of 2.

Algorithm D&C-MatMult(A, B)

1. $n \leftarrow$ number of rows of A
2. **if** $n = 1$ **then return** $(a_{11} b_{11})$
3. **else**
4. Let A_{ij}, B_{ij} (for $i, j = 1, 2$ be $(n/2 \times n/2)$-submatrices such that $$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \text{ and } B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$$
5. Recursively compute $A_{11}B_{11}$, $A_{12}B_{21}$, $A_{11}B_{12}$, $A_{12}B_{22}$, $A_{21}B_{11}$, $A_{22}B_{21}$, $A_{21}B_{12}$, $A_{22}B_{22}$
6. Compute $C_{11} = A_{11}B_{11} + A_{12}B_{21}$, $C_{12} = A_{11}B_{12} + A_{12}B_{22}$, $C_{21} = A_{21}B_{11} + A_{22}B_{21}$, $C_{22} = A_{21}B_{12} + A_{22}B_{22}$
7. **return** $\left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right)$

Analysis of D&C-MatMult

$T(n)$ is the number of operations done by D&C-MatMult.

- Lines $1, 2, 3, 4, 7$ require $\Theta(1)$ arithmetic operations
- Line 5 requires $8T(n/2)$ arithmetic operations
- Line 6 requires $4(n/2)^2 = \Theta(n^2)$ arithmetic operations.
  Remember! Size of matrices is $\Theta(n^2)$, NOT $\Theta(n)$

We get the recurrence

$$T(n) = 8T(n/2) + \Theta(n^2).$$

Since $\log_2(8) = 3$, the Master Theorem yields

$$T(n) = \Theta(n^3).$$

(No improvement over MatMult . . . why?)

Strassen's algorithm (1969)

Assume n is a power of 2.

Let

$$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \quad \text{and} \quad B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right).$$

We want to compute

$$\begin{aligned} AB &= \left(\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right) \\ &= \left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right). \end{aligned}$$

Strassen's algorithm uses a *trick* in applying Divide-and-Conquer.

Strassen's algorithm (cont'd)

Let

$$\begin{aligned}
P_1 &= (A_{11} + A_{22})(B_{11} + B_{22}) \\
P_2 &= (A_{21} + A_{22})B_{11} \\
P_3 &= A_{11}(B_{12} - B_{22}) \\
P_4 &= A_{22}(-B_{11} + B_{21}) && (*) \\
P_5 &= (A_{11} + A_{12})B_{22} \\
P_6 &= (-A_{11} + A_{21})(B_{11} + B_{12}) \\
P_7 &= (A_{12} - A_{22})(B_{21} + B_{22})
\end{aligned}$$

Then

$$\begin{aligned}
C_{11} &= P_1 + P_4 - P_5 + P_7 & C_{12} &= P_3 + P_5 \\
C_{21} &= P_2 + P_4 & C_{22} &= P_1 + P_3 - P_2 + P_6
\end{aligned}$$

$$(**)$$

Checking Strassen's algorithm - $C11$

We will check the equation for C_{11} is correct.

Strassen's algorithm computes $C_{11} = P1 + P4 - P5 + P7$. We have

$$
\begin{aligned}
P1 &= (A11 + A22)(B11 + B22) \\
&= A11B11 + A11B22 + A22B11 + A22B22. \\
P4 &= A22(-B11 + B21) \;=\; A22B21 - A22B11. \\
P5 &= (A11 + A12)B22 \;=\; A11B22 + A12B22. \\
P7 &= (A12 - A22)(B21 + B22) \\
&= A12B21 + A12B22 - A22B21 - A22B22.
\end{aligned}
$$

Then $P1 + P4 = A11B11 + A11B22 + A22B22 + A22B21$.

Then $P1 + P4 - P5 = A11B11 + A22B22 + A22B21 - A12B22$.

Then $P1 + P4 - P5 + P7 = A11B11 + A12B21$, which is $C11$.

Class exercise: check other 3 equations.

Strassen's algorithm (cont'd)

Crucial Observation

Only 7 multiplications of $(n/2 \times n/2)$-matrices are needed to compute AB.

Algorithm $\textsc{Strassen}(A, B)$

1. $n \leftarrow$ number of rows of A
2. **if** $n = 1$ **then return** $(a_{11} b_{11})$
3. **else**
4. Determine A_{ij} and B_{ij} for $i, j = 1, 2$ (as before)
5. Compute $P_1, \ldots, P_7$ as in $(*)$
6. Compute $C_{11}, C_{12}, C_{21}, C_{22}$ as in $(**)$
7. **return** $\left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right)$

Analysis of Strassen's algorithm

Let $T(n)$ be the number of arithmetic operations performed by STRASSEN.

- Lines $1-4$ and 7 require $\Theta(1)$ arithmetic operations
- Line 5 requires $7T(n/2)+\Theta(n^2)$ arithmetic operations
- Line 6 requires $\Theta(n^2)$ arithmetic operations. remember.

We get the recurrence

$$T(n) = 7T(n/2) + \Theta(n^2).$$

Since $\log_2(7) \approx 2.807 > 2$, the Master Theorem yields

$$T(n) = \Theta(n^{\log_2(7)}).$$

Remarks on matrix multiplication

- The current best (for asymptotic running time) algorithm is by Coppersmith & Winograd (1987), and has a running time of

$$\Theta(n^{2.376}).$$

- In practice, the "school" MatMult algorithm tends to outperform Strassen's algorithm, unless the matrices are huge.
- The best known lower bound for matrix multiplication is

$$\Omega(n^2).$$

  This is a *trivial* lower bound (need to look at all entries of each matrix). Amazingly, $\Omega(n^2)$ is believed to be "the truth"!
  Open problem: Can we find a $O(n^{2+o(1)})$-algorithm for Matrix Multiplication of $n \times n$ matrices?

Reading Assignment

[CLRS] Section 4.3 "Using the Master method" (pp. 73-75) and Section 28.2 (pp. 735-741). *Corresponds to Section 4.3 (pp. 61-63) and Section 31.2 (pp. 739-745) in [CLR].*

See Links from course webpage (for history).

Problems

1. Exercise 4.3-2, p. 75 of [CLRS]. *Ex 4.3-2, page 64 of [CLR].*
2. Exercise 28.2-1, p. 741 of [CLRS]. *Ex 31.2-1, page 744 of [CLR].*
3. On page 5, I state that the "school" algorithm MATMULT has running time $\Theta(n^3)$. The $O(n^3)$ is fairly easy to see (I think! If not, ask me).
   Show the $\Omega(n^3)$ bound for MATMULT.