

# Algorithms and Data Structures: Counting sort and Radix sort

18th October, 2011

## Special Cases of the Sorting Problem

In this lecture we assume that the sort keys are sequences of bits.

- ▶ Quite a natural special case. Doesn't cover everything:
  - ▶ eg, exact real number arithmetic doesn't take this form.
  - ▶ In certain applications, eg Biology, pairwise experiments may only return  $>$  or  $<$  (non-numeric).
- ▶ *Sometimes* the bits are naturally grouped, eg, as characters in a string or hexadecimal digits in a number (4 bits), or in general bytes (8 bits).
- ▶ **Today's** sorting algorithms are allowed access these bits or groups of bits, instead of just letting them compare keys ...  
This was NOT possible in comparison-based setting.

## Easy results ... Surprising results

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### Surprising case: (I think)

For any constant  $k$ , the problem of sorting  $n$  integers in the range  $\{1, \dots, n^k\}$  can be done in  $\Theta(n)$  time.

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$C[j] =$  the number of keys *less than or equal* to  $j$ .

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Then we know that the elements with key “ $j$ ” must be stored at the indices  $C[j - 1] + 1, \dots, C[j]$  of the final sorted array.
3. We use a “trick” to move the elements to the right position of an auxiliary array. Then we copy the sorted auxiliary array back to the original one.

## Implementation of Counting Sort

**Algorithm** COUNTING SORT( $A, m$ )

1.  $n \leftarrow A.length$
2. Initialise array  $C[1 \dots m]$
3. **for**  $i \leftarrow 1$  **to**  $n$  **do**
4.      $j \leftarrow A[i].key$
5.      $C[j] \leftarrow C[j] + 1$
6. **for**  $j \leftarrow 2$  **to**  $m$  **do**
7.      $C[j] \leftarrow C[j] + C[j - 1]$     $\triangleright$   $C[j]$  stores # of keys  $\leq j$
8. Initialise array  $B[1 \dots n]$
9. **for**  $i \leftarrow n$  **downto**  $1$  **do**
10.      $j \leftarrow A[i].key$       $\triangleright$   $A[i]$  highest w. key  $j$
11.      $B[C[j]] \leftarrow A[i]$       $\triangleright$  Insert  $A[i]$  into highest free index for  $j$
12.      $C[j] \leftarrow C[j] - 1$
13. **for**  $i \leftarrow 1$  **to**  $n$  **do**
14.      $A[i] \leftarrow B[i]$

## Analysis of Counting Sort

- ▶ The loops in lines 3–5, 9–12, and 13–14 all require time  $\Theta(n)$ .
- ▶ The loop in lines 6–7 requires time  $\Theta(m)$ .
- ▶ Thus the overall running time is

$$O(n + m).$$

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Note: COUNTING-SORT is STABLE.

- ▶ (After sorting, 2 items with the same key have their *initial relative order*).

# Radix Sort

## Basic Assumption

Keys are sequences of **digits** in a fixed range  $0, \dots, R - 1$ , all of equal length  $d$ .

## Examples of such keys

- ▶ 4 digit hexadecimal numbers (corresponding to 16 bit integers)  
 $R = 16, d = 4$
- ▶ 5 digit decimal numbers (for example, US post codes)  
 $R = 10, d = 5$
- ▶ Fixed length ASCII character sequences  
 $R = 128$
- ▶ Fixed length byte sequences  
 $R = 256$

# Stable Sorting Algorithms

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## Examples

- ▶ COUNTING-SORT, MERGE-SORT, and INSERTION SORT are all stable.
- ▶ QUICKSORT is **not** stable.
- ▶ If keys and elements are exactly the same thing (in our setting, an element is a structure containing the key as a sub-element) then we have a much easier (non-stable) version of COUNTING-SORT. (How? ... **CLASS?**).

## Radix Sort (cont'd)

### Idea

Sort the keys digit by digit, *starting with the least significant digit.*

### Example

now	sob	tag	ace
for	nob	ace	bet
tip	ace	bet	dim
ilk	tag	dim	for
dim	ilk	tip	hut
dim	dim	sky	ilk
tag	tip	ilk	jot
jot	for	sob	nob
sob	jot	nob	now
nob	hut	for	sky
sky	bet	jot	sob
hut	now	now	tag
ace	sky	hut	tip
bet			

## Radix Sort (cont'd)

**Algorithm** RADIX-SORT( $A, d$ )

1. **for**  $i \leftarrow 0$  **to**  $d$  **do**
2.       use stable sort to sort array  $A$  using digit  $i$  as key

Most commonly, COUNTING SORT is used in line 2 - this means that once a set of digits is already in sorted order, then (by **stability**) performing COUNTING SORT on the *next-most significant* digits preserves that order, within the “blocks” constructed by the new iteration.

Then each execution of line 2 requires time  $\Theta(n + R)$ .

Thus the overall time required by RADIX-SORT is

$$\Theta(d(n + R))$$

## Sorting Integers with Radix-Sort

### Theorem 2

An array of length  $n$  whose keys are  $b$ -bit numbers can be sorted in time

$$\Theta(n \lceil b / \lg n \rceil)$$

using a suitable version of RADIX-SORT.

**Proof:** Let the digits be blocks of  $\lceil \lg n \rceil$  bits. Then  $R = 2^{\lceil \lg n \rceil} = \Theta(n)$  and  $d = \lceil b / \lceil \lg n \rceil \rceil$ . Using the implementation of RADIX-SORT based on COUNTING SORT the integers can be sorted in time

$$\Theta(d(n + R)) = \Theta(n \lceil b / \lg n \rceil).$$

Note: If all numbers are at most  $n^k$ , then  $b = k \lg n \dots \Rightarrow$  Radix Sort is  $\Theta(n)$  (assuming  $k$  is some constant, eg 3, 10).

## Reading Assignment

[CLRS] Sections 8.2, 8.3 or

[CLR] Sections 9.1–9.3

### Problems

1. Think about the qn. on slide 7 - how do we get a very easy (non-stable) version of COUNTING-SORT if there are no items attached to the keys?
2. Exercise 8.3-4 of [CLRS]. *This is 9.3-4 of [CLR].*