Algorithms and Data Structures:
Counting sort and Radix sort

14th October, 2014
Special Cases of the Sorting Problem

In this lecture we assume that the sort keys are sequences of bits.

- Quite a natural special case. Doesn’t cover everything:
  - eg, exact real number arithmetic doesn’t take this form.
  - In certain applications, eg Biology, pairwise experiments may only return $>$ or $<$ (non-numeric).

- Sometimes the bits are naturally grouped, eq, as characters in a string or hexadecimal digits in a number (4 bits), or in general bytes (8 bits).

- Today’s sorting algorithms are allowed access these bits or groups of bits, instead of just letting them compare keys . . .
  This was NOT possible in comparison-based setting.
Easy results . . . Surprising results

Simplest Case:
Keys are integers in the range 1, . . . , m, where $m = O(n)$ ($n$ is (as usual) the number of elements to be sorted). We can sort in $\Theta(n)$ time.
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Surprising case: (I think)
For any constant $k$, the problem of sorting $n$ integers in the range $\{1, \ldots, n^k\}$ can be done in $\Theta(n)$ time.
Counting Sort

Assumption: Keys (attached to items) are Ints in range 1, ..., \( m \).

Idea
1. Count for every key \( j \), \( 1 \leq j \leq m \) how often it occurs in the input array. Store results in an array \( C \).
2. The counting information stored in \( C \) can be used to determine the position of each element in the sorted array. Suppose we modify the values of the \( C[j] \) so that now \( C[j] = \) the number of keys less than or equal to \( j \). Then we know that the elements with key "\( j \)" must be stored at the indices \( C[j-1] + 1, \ldots, C[j] \) of the final sorted array.
3. We use a "trick" to move the elements to the right position of an auxiliary array. Then we copy the sorted auxiliary array back to the original one.
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1. Count for every key $j$, $1 \leq j \leq m$ how often it occurs in the input array. Store results in an array $C$.

2. The counting information stored in $C$ can be used to determine the position of each element in the sorted array. Suppose we modify the values of the $C[j]$ so that now

   $C[j] =$ the number of keys less than or equal to $j$.

   Then we know that the elements with key “$j$” must be stored at the indices $C[j - 1] + 1, \ldots, C[j]$ of the final sorted array.

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3. We use a “trick” to move the elements to the right position of an auxiliary array. Then we copy the sorted auxiliary array back to the original one.
Algorithm **Counting Sort**$(A, m)$
1. $n \leftarrow A$.length
2. Initialise array $C[1 \ldots m]$
3. for $i \leftarrow 1$ to $n$ do
4. $j \leftarrow A[i]$.key
5. $C[j] \leftarrow C[j] + 1$
6. for $j \leftarrow 2$ to $m$ do
7. $C[j] \leftarrow C[j] + C[j - 1]$ \Comment{C[j] stores \# of keys $\leq j$}
8. Initialise array $B[1 \ldots n]$
9. for $i \leftarrow n$ downto $1$ do
10. $j \leftarrow A[i]$.key \Comment{$A[i]$ highest w. key $j$}
11. $B[C[j]] \leftarrow A[i]$ \Comment{Insert $A[i]$ into highest free index for $j$}
12. $C[j] \leftarrow C[j] - 1$
13. for $i \leftarrow 1$ to $n$ do

*ADS: lect 8 – slide 5 – 14th October, 2014*
Analysis of Counting Sort

- The loops in lines 3–5, 9–12, and 13–14 all require time $\Theta(n)$.
- The loop in lines 6–7 requires time $\Theta(m)$.
- Thus the overall running time is

$$O(n + m).$$

- This is linear in the number of elements if $m = O(n)$.
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Note: This does not contradict Theorem 3 from Lecture 7 - that’s a result about the general case, where keys have an arbitrary size (and need not even be numeric).
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Note: **Counting-Sort** is STABLE.

- *(After sorting, 2 items with the same key have their *initial relative order).*

*ADS: lect 8 – slide 6 – 14th October, 2014*
Radix Sort

Basic Assumption

Keys are sequences of digits in a fixed range $0, \ldots, R - 1$, all of equal length $d$.

Examples of such keys

- 4 digit hexadecimal numbers (corresponding to 16 bit integers)
  $R = 16, d = 4$

- 5 digit decimal numbers (for example, US post codes)
  $R = 10, d = 5$

- Fixed length ASCII character sequences
  $R = 128$

- Fixed length byte sequences
  $R = 256$
Stable Sorting Algorithms

Definition 1
A sorting algorithm is stable if it always leaves elements with equal keys in their original order.
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Examples
- **Counting-Sort, Merge-Sort, and Insertion Sort** are all stable.
- **Quicksort** is not stable.
- If keys and elements are exactly the same thing (in our setting, an element is a structure containing the key as a sub-element) then we have a much easier (non-stable) version of **Counting-Sort**. (How? ... CLASS?).
Radix Sort (cont’d)

Idea

Sort the keys digit by digit, *starting with the least significant digit.*

Example

<table>
<thead>
<tr>
<th>now</th>
<th>sob</th>
<th>tag</th>
<th>ace</th>
</tr>
</thead>
<tbody>
<tr>
<td>for</td>
<td>nob</td>
<td>ace</td>
<td>bet</td>
</tr>
<tr>
<td>tip</td>
<td>ace</td>
<td>bet</td>
<td>dim</td>
</tr>
<tr>
<td>ilk</td>
<td>tag</td>
<td>bet</td>
<td>dim</td>
</tr>
<tr>
<td>dim</td>
<td>ilk</td>
<td>tip</td>
<td>hut</td>
</tr>
<tr>
<td>tag</td>
<td>dim</td>
<td>sky</td>
<td>ilk</td>
</tr>
<tr>
<td>jot</td>
<td>tip</td>
<td>ilk</td>
<td>jot</td>
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<tr>
<td>sob</td>
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<td>now</td>
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<tr>
<td>sky</td>
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</tr>
<tr>
<td>hut</td>
<td>bet</td>
<td>jot</td>
<td>sob</td>
</tr>
<tr>
<td>ace</td>
<td>now</td>
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<td>tip</td>
</tr>
</tbody>
</table>

Each of the three sorts is carried out with respect to the digits in that column. “Stability” (and having previously sorted digits/suffixes to the right), means this achieves a sorting of the suffixes starting at the current column.

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Radix Sort (cont’d)

Algorithm \texttt{Radix-Sort}(A, d)
\begin{enumerate}
  \item for \(i \leftarrow 0\) to \(d\) do
  \item use stable sort to sort array \(A\) using digit \(i\) as key
\end{enumerate}

Most commonly, \texttt{Counting Sort} is used in line 2 - this means that once a set of digits is already in sorted order, then (by stability) performing \texttt{Counting Sort} on the \textit{next-most significant} digits preserves that order, within the “blocks” constructed by the new iteration.

Then each execution of line 2 requires time \(\Theta(n + R)\).
Thus the overall time required by \texttt{Radix-Sort} is
\[\Theta(d(n + R))\]
Theorem 2
An array of length $n$ whose keys are $b$-bit numbers can be sorted in time

$$\Theta(n\lceil b/\lg n \rceil)$$

using a suitable version of Radix-Sort.

Proof: Let the digits be blocks of $[\lg n]$ bits. Then $R = 2^{\lceil \lg n \rceil} = \Theta(n)$ and $d = \lceil b/\lceil \lg n \rceil \rceil$. Using the implementation of Radix-Sort based on Counting Sort the integers can be sorted in time

$$\Theta(d(n + R)) = \Theta(n\lceil b/\lg n \rceil).$$

Note: If all numbers are at most $n^k$, then $b = k \lg n \ldots \Rightarrow$ Radix Sort is $\Theta(n)$ (assuming $k$ is some constant, eg 3, 10).
Reading Assignment

[CLRS] Sections 8.2, 8.3

Problems

1. Think about the qn. on slide 7 - how do we get a very easy (non-stable) version of COUNTING-SORT if there are no items attached to the keys?

2. Can you come up with another way of achieving counting sort’s $O(m + n)$-time bound and stability (you will need a different data structure from an array).

3. Exercise 8.3-4 of [CLRS].