

Algorithms and Data Structures: Counting sort and Radix sort

18th October, 2011

ADS: lect 8 – slide 1 – 18th October, 2011

Easy results ... Surprising results

Simplest Case:

Keys are integers in the range $1, \dots, m$, where $m = O(n)$ (n is (as usual) the number of elements to be sorted). We can sort in $\Theta(n)$ time (big deal ... but will help later).

Surprising case: (I think)

For any constant k , the problem of sorting n integers in the range $\{1, \dots, n^k\}$ can be done in $\Theta(n)$ time.

ADS: lect 8 – slide 3 – 18th October, 2011

Special Cases of the Sorting Problem

In this lecture we assume that the sort keys are sequences of bits.

- ▶ Quite a natural special case. Doesn't cover everything:
 - ▶ eg, exact real number arithmetic doesn't take this form.
 - ▶ In certain applications, eg Biology, pairwise experiments may only return $>$ or $<$ (non-numeric).
- ▶ *Sometimes* the bits are naturally grouped, eg, as characters in a string or hexadecimal digits in a number (4 bits), or in general bytes (8 bits).
- ▶ *Today's* sorting algorithms are allowed access these bits or groups of bits, instead of just letting them compare keys ... This was NOT possible in comparison-based setting.

ADS: lect 8 – slide 2 – 18th October, 2011

Counting Sort

Assumption: Keys (attached to items) are ints in range $1, \dots, m$.

Idea

1. Count for every key j , $1 \leq j \leq m$ how often it occurs in the input array. Store results in an array C .
2. The counting information stored in C can be used to determine the position of each element in the sorted array. Suppose we modify the values of the $C[j]$ so that *now*
 $C[j] =$ the number of keys *less than or equal to* j .
Then we know that the elements with key " j " must be stored at the indices $C[j-1] + 1, \dots, C[j]$ of the final sorted array.
3. We use a "trick" to move the elements to the right position of an auxiliary array. Then we copy the sorted auxiliary array back to the original one.

ADS: lect 8 – slide 4 – 18th October, 2011

Implementation of Counting Sort

Algorithm COUNTING SORT(A, m)

1. $n \leftarrow A.length$
2. Initialise array $C[1 \dots m]$
3. **for** $i \leftarrow 1$ **to** n **do**
4. $j \leftarrow A[i].key$
5. $C[j] \leftarrow C[j] + 1$
6. **for** $j \leftarrow 2$ **to** m **do**
7. $C[j] \leftarrow C[j] + C[j - 1]$ ▷ $C[j]$ stores # of keys $\leq j$
8. Initialise array $B[1 \dots n]$
9. **for** $i \leftarrow n$ **downto** 1 **do**
10. $j \leftarrow A[i].key$ ▷ $A[i]$ highest w. key j
11. $B[C[j]] \leftarrow A[i]$ ▷ Insert $A[i]$ into highest free index for j
12. $C[j] \leftarrow C[j] - 1$
13. **for** $i \leftarrow 1$ **to** n **do**
14. $A[i] \leftarrow B[i]$

ADS: lect 8 – slide 5 – 18th October, 2011

Radix Sort

Basic Assumption

Keys are sequences of **digits** in a fixed range $0, \dots, R - 1$, all of equal length d .

Examples of such keys

- ▶ 4 digit hexadecimal numbers (corresponding to 16 bit integers)
 $R = 16, d = 4$
- ▶ 5 digit decimal numbers (for example, US post codes)
 $R = 10, d = 5$
- ▶ Fixed length ASCII character sequences
 $R = 128$
- ▶ Fixed length byte sequences
 $R = 256$

ADS: lect 8 – slide 7 – 18th October, 2011

Analysis of Counting Sort

- ▶ The loops in lines 3–5, 9–12, and 13–14 all require time $\Theta(n)$.
- ▶ The loop in lines 6–7 requires time $\Theta(m)$.
- ▶ Thus the overall running time is

$$O(n + m).$$

- ▶ This is *linear* in the number of elements if $m = O(n)$.

Note: This does not contradict Theorem 3 from Lecture 7 - that's a result about the **general case**, where keys have an arbitrary size (and need not even be numeric).

Note: COUNTING-SORT is STABLE.

- ▶ (After sorting, 2 items with the same key have their *initial relative order*).

ADS: lect 8 – slide 6 – 18th October, 2011

Stable Sorting Algorithms

Definition 1

A sorting algorithm is **stable** if it always leaves elements with equal keys in their original order.

Examples

- ▶ COUNTING-SORT, MERGE-SORT, and INSERTION SORT are all stable.
- ▶ QUICKSORT is **not** stable.
- ▶ If keys and elements are exactly the same thing (in our setting, an element is a structure containing the key as a sub-element) then we have a much easier (non-stable) version of COUNTING-SORT. (How? ... CLASS?).

ADS: lect 8 – slide 8 – 18th October, 2011

Radix Sort (cont'd)

Idea

Sort the keys digit by digit, *starting with the least significant digit.*

Example

now	sob	tag	ace
for	nob	ace	bet
tip	ace	bet	dim
ilk	tag	dim	for
dim	ilk	tip	hut
tag	dim	sky	ilk
jot	tip	ilk	jot
sob	for	sob	nob
nob	jot	nob	now
sky	hut	for	sky
hut	bet	jot	sob
ace	now	now	tag
bet	sky	hut	tip

ADS: lect 8 – slide 9 – 18th October, 2011

Sorting Integers with Radix-Sort

Theorem 2

An array of length n whose keys are b -bit numbers can be sorted in time

$$\Theta(n \lceil b / \lg n \rceil)$$

using a suitable version of RADIX-SORT.

Proof: Let the digits be blocks of $\lceil \lg n \rceil$ bits. Then $R = 2^{\lceil \lg n \rceil} = \Theta(n)$ and $d = \lceil b / \lceil \lg n \rceil \rceil$. Using the implementation of RADIX-SORT based on COUNTING SORT the integers can be sorted in time

$$\Theta(d(n + R)) = \Theta(n \lceil b / \lg n \rceil).$$

Note: If all numbers are at most n^k , then $b = k \lg n \dots \Rightarrow$ Radix Sort is $\Theta(n)$ (assuming k is some constant, eg 3, 10).

ADS: lect 8 – slide 11 – 18th October, 2011

Radix Sort (cont'd)

Algorithm RADIX-SORT(A, d)

1. **for** $i \leftarrow 0$ **to** d **do**
2. use stable sort to sort array A using digit i as key

Most commonly, COUNTING SORT is used in line 2 - this means that once a set of digits is already in sorted order, then (by **stability**) performing COUNTING SORT on the *next-most significant* digits preserves that order, within the “blocks” constructed by the new iteration.

Then each execution of line 2 requires time $\Theta(n + R)$.

Thus the overall time required by RADIX-SORT is

$$\Theta(d(n + R))$$

ADS: lect 8 – slide 10 – 18th October, 2011

Reading Assignment

[CLRS] Sections 8.2, 8.3 or
[CLR] Sections 9.1–9.3

Problems

1. Think about the qn. on slide 7 - how do we get a very easy (non-stable) version of COUNTING-SORT if there are no items attached to the keys?
2. Exercise 8.3-4 of [CLRS]. *This is 9.3-4 of [CLR].*

ADS: lect 8 – slide 12 – 18th October, 2011