Algorithms and Data Structures: Lower Bounds for Sorting

1st February, 2016

Comparison Based Sorting Algorithms

Definition 1

A sorting algorithm is *comparison based* if comparisons $A[i] < A[j]$, $A[i] \leq A[j]$, $A[i] = A[j]$, $A[i] \geq A[j]$, $A[i] > A[j]$ are the only ways in which it accesses the input elements.

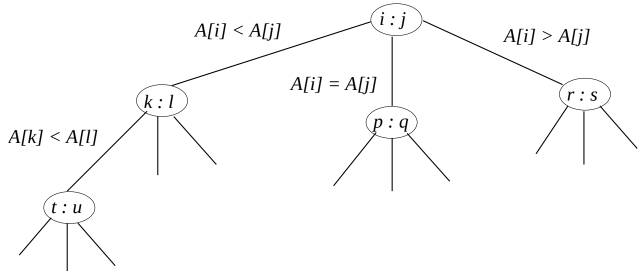
Comparison based sorting algorithms are *generic* in the sense that they can be used for all types of elements that are *comparable* (such as objects of type `Comparable` in Java).

Example 2

INSERTION-SORT, QUICKSORT, MERGE-SORT, HEAPSORT are all comparison based.

The Decision Tree Model

Abstractly, we may describe the behaviour of a comparison-based sorting algorithm S on an input array $A = \langle A[1], \dots, A[n] \rangle$ by a decision tree:

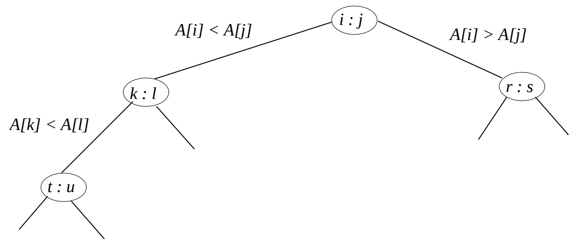


At each leaf of the tree the output of the algorithm on the corresponding execution branch will be displayed. Outputs of sorting algorithms correspond to permutations of the input array.

A Simplifying Assumption

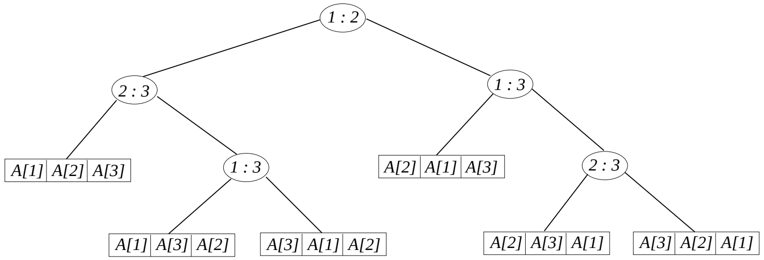
In the following, we assume that all keys of elements of the input array of a sorting algorithm are distinct. (It is ok to restrict to a special case, because we want to prove a lower bound.)

Thus the outcome $A[i] = A[j]$ in a comparison will never occur, and the decision tree is in fact a binary tree:



Example

Insertion sort for $n = 3$:



In insertion sort, when we get the result of a comparison, we often swap some elements of the array. In showing decision trees, we don't *implement* a swap. Our indices always refer to the *original* elements at that position in the array. To understand what I mean, draw the evolving array of InsertionSort beside this decision tree.

A Lower Bound for Comparison Based Sorting

For a comparison based sorting algorithm S:

$$C_S(n) = \text{worst-case number of comparisons performed by } S \text{ on an input array of size } n.$$

Theorem 3

For all *comparison based sorting algorithms S we have*

$$C_S(n) = \Omega(n \lg n).$$

Corollary 4

The worst-case running time of any comparison based sorting algorithm is $\Omega(n \lg n)$.

A Lower Bound for Comparison Based Sorting

Proof of Theorem 3 uses Decision-Tree Model of sorting.

It is an Information-Theoretic Lower Bound:

- "Information-Theoretic" means that it is based on the amount of "information" that an instance of the problem can encode.
- For sorting, the input can encode $n!$ outputs.
- Proof does not make *any* assumption about *how* the sorting might be done (except it is comparison-based).

Proof of Theorem 3

Observation 5

For every n, $C_S(n)$ is the height of the decision tree of S on inputs n (the longest path from the "root" to a leaf is the maximum number of comparisons that algorithm S will do on an input of length n).

We shall prove a lower bound for the height of the decision tree for *any* algorithm S.

Remark

Maybe you are wondering ... was it really ok to assume all keys are distinct?

It is ok - because the problem of sorting n keys (with no distinctness assumption) is *more general* than the problem of sorting n distinct keys.

The worst-case for sorting certainly is as bad as the worst-case for all-distinct keys sorting.

Observation 6
Each permutation of the inputs must occur at at least one *leaf of the decision tree.*

(Obs 6 *must* be true, if our algorithm is to sort properly for *all* inputs.).

- By Obs 6, the decision tree on inputs of size n has at least $n!$ leaves (for *any* algorithm S).
- The (simplified) decision tree is a binary tree. A binary tree of height h has at most 2^h leaves.
- Putting everything together, we get

$$\begin{aligned} n! &\leq \text{number of leaves of decision tree} \\ &\leq 2^{\text{height of decision tree}} \\ &\leq 2^{C_S(n)}. \end{aligned}$$

- Thus

$$C_S(n) \geq \lg(n!) = \Omega(n \lg(n)).$$

To obtain the last inequality, we can use the following inequality:

$$n^{n/2} \leq n! \leq n^n$$

This tells us that $\lg n! \geq \lg(n^{n/2}) = (n/2)\lg n = \Omega(n \lg(n))$.

Thm 3 QED

An Average Case Lower Bound

For any comparison based sorting algorithm S:

$A_S(n)$ = average number of comparisons performed by S on an input array of size n.

Theorem 7
For all comparison based sorting algorithms S we have

$$A_S(n) = \Omega(n \lg n).$$

Proof uses the fact that the average length of a path from the root to a leaf in a binary tree with ℓ leaves is $\Omega(\lg \ell)$ (Theorem 11 and 12).

Corollary 8
The average-case running time of any comparison based sorting algorithm is $\Omega(n \lg n)$.

Average root-leaf length in Binary tree

Definition 9
For any binary tree T, let $AvgRL(T)$ denote the *average root-to-leaf* length for T.

Definition 10
A *near-complete* binary tree T is a binary tree in which every internal node has *exactly* two child nodes, and all leaves are either at depth h or depth $h-1$.

Theorem 11
Any "near-complete" binary tree T with leaf set $L(T)$, $|L(T)| \geq 4$, has Average root-to-leaf *length $AvgRL(T)$* at least *$lg(|L(T)|)/2$.*

Theorem 12
For any binary tree T, there is a near-complete binary tree T' such that $L(T) = L(T')$ (same leaf set) and such that $AvgRL(T') \leq AvgRL(T)$. Hence $AvgRL(T) \geq \lg(|L(T)|)/2$ holds for all binary trees.

Proof of Theorems 11 and 12 via BOARD notes.

Implications of These Lower Bounds

Theorem 3 and Theorem 7 are significant because they hold for *all* comparison-based algorithms S. They imply the following:

1. By Thm 3, any comparison-based algorithm for sorting which has a worst-case running-time of $O(n \lg n)$ is *asymptotically optimal* (ie, apart from the constant factor inside the "O" term, it is as good as possible in terms of worst-case analysis). This includes algorithms like MergeSort, HeapSort.
2. By Thm 7, any comparison-based algorithm for sorting which has an average-case running-time of $O(n \lg n)$ is the best you can hope for in terms of average-case analysis (apart from the constant factor inside the "O" term). This is accomplished by MergeSort and HeapSort. In Lecture 7, we will see it is also true for QuickSort.

Lecture 9 (after average-case analysis of QuickSort)

We will show how in a *special case* of sorting (when the inputs are numbers, coming from the range $\{1, 2, \ldots, n^k\}$ for some constant k, we can sort in linear time (NOT a comparison-based algorithm).

Reading Assignment

[CLRS] Section 8.1 (2nd and 3rd edition) or
[CLR] Section 9.1
Well-worth reading - this is a nice chapter of CLRS (not too long).

Problems

1. Draw (simplified) decision trees for INSERTION SORT and QUICKSORT for $n = 4$.
2. Exercise 8.1-1 of [CLRS] (both 2nd and 3rd ed).
3. Resolve the complexity (in terms of no-of-comparisons) of sorting 4 numbers.
   3.1 Give an algorithm which sorts any 4 numbers and which uses at most 5 comparisons in the worst-case.
   3.2 Prove (using the decision-tree model) that there is no algorithm to sort 4 numbers, which uses less than 5 comparisons in the worst-case.