Algorithms and Data Structures: Counting sort and Radix sort

22nd October, 2010
Special Cases of the Sorting Problem

In this lecture we assume that the sort keys are sequences of bits.

- Quite a natural special case. Doesn’t cover everything:
  - eg, exact real number arithmetic doesn’t take this form.
  - In certain applications, eg Biology, pairwise experiments may only return $>$ or $<$ (non-numeric).

- Sometimes the bits are naturally grouped, eq, as characters in a string or hexadecimal digits in a number (4 bits), or in general bytes (8 bits).

- Today’s sorting algorithms are allowed access these bits or groups of bits, instead of just letting them compare keys . . .
  This was NOT possible in comparison-based setting.
Easy results . . . Surprising results

Simplest Case:
Keys are integers in the range 1, . . . , $m$, where $m = O(n)$ ($n$ is (as usual) the number of elements to be sorted). We can sort in $\Theta(n)$ time.
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Surprising case: (I think)
For any constant \( k \), the problem of sorting \( n \) integers in the range \( \{1, \ldots, n^k\} \) can be done in \( \Theta(n) \) time.
Counting Sort

Assumption: Keys (attached to items) are Ints in range 1, . . . , m.

Idea
1. Count for every key $j$, $1 \leq j \leq m$ how often it occurs in the input array. Store results in an array $C$.
2. The counting information stored in $C$ can be used to determine the position of each element in the sorted array. Suppose we modify the values of the $C[j]$ so that $C[j] =$ the number of keys less than or equal to $j$. Then we know that the elements with key $j$ must be stored at the indices $C[j] - 1 + 1$, . . . , $C[j]$ of the final sorted array.
3. We use a “trick” to move the elements to the right position of an auxiliary array. Then we copy the sorted auxiliary array back to the original one.
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   \[ C[j] = \text{the number of keys less than or equal to } j. \]
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Implementation of Counting Sort

Algorithm Counting Sort(A, m)
1. \( n \leftarrow A\text{.length} \)
2. Initialise array \( C[1 \ldots m] \)
3. for \( i \leftarrow 1 \) to \( n \) do
4. \( j \leftarrow A[i]\text{.key} \)
5. \( C[j] \leftarrow C[j] + 1 \)
6. for \( j \leftarrow 2 \) to \( m \) do
7. \( C[j] \leftarrow C[j] + C[j - 1] \quad \triangleright \text{C[j] stores \# of keys} \leq j \)
8. Initialise array \( B[1 \ldots n] \)
9. for \( i \leftarrow n \) downto \( 1 \) do
10. \( j \leftarrow A[i]\text{.key} \quad \triangleright \text{A[i] highest w. key} j \)
11. \( B[C[j]] \leftarrow A[i] \quad \triangleright \text{Insert A[i] into highest free index for} j \)
12. \( C[j] \leftarrow C[j] - 1 \)
13. for \( i \leftarrow 1 \) to \( n \) do
14. \( A[i] \leftarrow B[i] \)
Analysis of Counting Sort

- The loops in lines 3–5, 9–12, and 13–14 all require time $\Theta(n)$.
- The loop in lines 6–7 requires time $\Theta(m)$.
- Thus the overall running time is

$$O(n + m).$$

- This is linear in the number of elements if $m = O(n)$. 

Note: This does not contradict Theorem 3 from Lecture 7 - that's a result about the general case, where keys have an arbitrary size (and need not even be numeric). 

Note: Counting-Sort is stable. 

(After sorting, 2 items with the same key have their initial relative order.)
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Radix Sort

Basic Assumption

Keys are sequences of digits in a fixed range 0, \ldots, R − 1, all of equal length \(d\).

Examples of such keys

- 4 digit hexadecimal numbers (corresponding to 16 bit integers)
  \(R = 16, d = 4\)

- 5 digit decimal numbers (for example, US post codes)
  \(R = 10, d = 5\)

- Fixed length ASCII character sequences
  \(R = 128\)

- Fixed length byte sequences
  \(R = 256\)
Stable Sorting Algorithms

Definition 1
A sorting algorithm is stable if it always leaves elements with equal keys in their original order.
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Examples
- **Counting-Sort, Merge-Sort, and Insertion Sort** are all stable.
- **Quicksort** is not stable.
- If keys and elements are exactly the same thing (in our setting, an element is a structure containing the key as a sub-element) then we have a much easier (non-stable) version of **Counting-Sort**. (How? ... CLASS?).
Radix Sort (cont’d)

Idea
Sort the keys digit by digit, *starting with the least significant digit*.

Example

```
now for  sob  tag  ace  ace  bet
for nob  ace  bet  bet  dim  bet
tip        tag  dim  for  hut  hut
ilk        ilk  tip  tip  sky  sky
dim        dim  ilk  for  sky  sky
   tag  tip  ilk  for  sky  sky
  jot  for  sob  nob  now  now
sob  jot  nob  now  now  tag
nob        sob  nob  now  now  tag
   sky  hut  bet  jot  sob  sob
   hut  bet  jot  sob  sob
  ace  now  now  tag  tag  tag
  bet  sky  hut  hut  hut  tip
```
Radix Sort (cont’d)

Algorithm Radix-Sort(A, d)
1. for i ← 0 to d do
2. use stable sort to sort array A using digit i as key

Most commonly, Counting Sort is used in line 2 - this means that once a set of digits is already in sorted order, then (by stability) performing Counting Sort on the next-most significant digits preserves that order, within the “blocks” constructed by the new iteration.

Then each execution of line 2 requires time $\Theta(n + R)$. Thus the overall time required by Radix-Sort is

$\Theta(d(n + R))$
Sorting Integers with Radix-Sort

Theorem 2
An array of length \( n \) whose keys are \( b \)-bit numbers can be sorted in time

\[
\Theta(n\lceil b/\lg n \rceil)
\]

using a suitable version of Radix-Sort.

Proof: Let the digits be blocks of \( \lceil \lg n \rceil \) bits. Then \( R = 2^{\lceil \lg n \rceil} = \Theta(n) \) and \( d = \lceil b/\lceil \lg n \rceil \rceil \). Using the implementation of Radix-Sort based on Counting Sort the integers can be sorted in time

\[
\Theta(d(n + R)) = \Theta(n\lceil b/\lg n \rceil)
\]

Note: If all numbers are at most \( n^k \), then \( b = k \lg n \) \( \Rightarrow \) Radix Sort is \( \Theta(n) \) (assuming \( k \) is some constant, eg 3, 10).
Reading Assignment

[CLRS] Sections 8.2, 8.3 or
[CLR] Sections 9.1–9.3

Problems

1. Think about the qn. on slide 7 - how do we get a very easy (non-stable) version of COUNTING-SORT if there are no items attached to the keys?
2. Exercise 8.3-4 of [CLRS]. This is 9.3-4 of [CLR].