Minimum Spanning Tree lectures $\boxed{\text{BOARD NOTE}}$

proof of correctness of Kruskal's Algorithm (slide 23)

proof of Step 3: (the interesting bit)

$\boxed{\text{CLAIM}}$ At any stage of "Kruskal", the current forest (V,F) is contained in some MST T of the graph G.

proof by induction

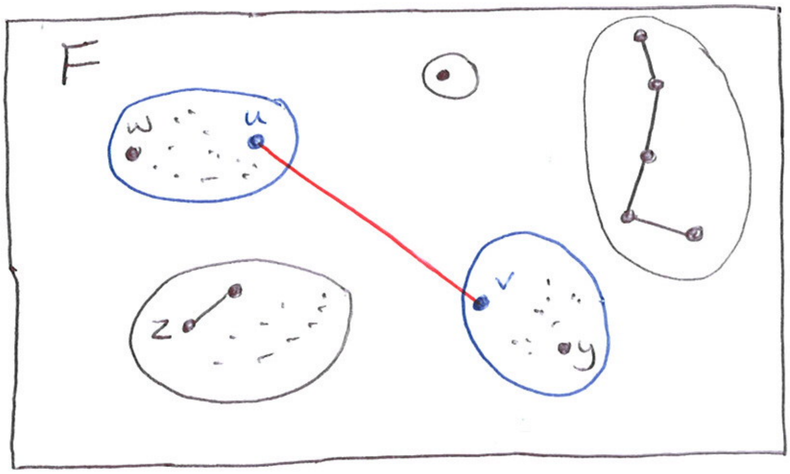
base case: Initially each component is an isolated node, ie, $F = \{\}$. Then $F \subseteq E(T)$ trivially holds for any MST T.

induction step:
Suppose we have (V,F) s.th. $F \subseteq E(T)$ for the MST T.
Suppose the next edge added by 'Kruskal' is $e=(u,v)$.
We need to show $F \cup \{e\}$ will then be contained in some MST of G (does not have to be the same MST T we have been thinking of till now)

case(a) Maybe e was in T anyway. Then we are done. (for induction step) ✓

case(b) Suppose $e \notin T$.
Then we have the situation on the right

[black edges are edges in F (and hence also in $E(T)$)
(u,v) is NOT in T.]



Now, since $e=(u,v)$ is NOT in T, consider the path p_{uv} between u and v in T.

Since u and v belong to different components of F, the path p_{uv} must contain some edge $f=(u^*,v^*)$ which is not in F and where u^* and v^* are in different components of F.

(e.g. in picture, might have $f=(z,w)$ or $f=(y,u)$ say)

f has not been added to F, so (as its endpoints are in different components) it must be that $w(f) \geq w(e)$ (*)
(otherwise Kruskal would already have added it)

Take $T \setminus \{f\} \Rightarrow$ two subtrees T_{u^*} and T_{v^*}. There is no path between u and v any more, so $(T\setminus\{f\}) \cup \{e\}$ is again a tree.
Also by (*) $w((T\setminus\{f\})\cup\{e\}) \leq w(T)$.
So $(T\setminus\{f\})\cup\{e\}$ is a MST satisfying $\boxed{\text{CLAIM}}$ for $F\cup\{e\}$, as required ////