Algorithms and Data Structures: Counting sort and Radix sort

Special Cases of the Sorting Problem

In this lecture we assume that the sort keys are sequences of bits.

- Quite a natural special case. Doesn't cover everything:
  - e.g., exact real number arithmetic doesn't take this form.
  - In certain applications, e.g. in Biology, pairwise experiments may only return $>$ or $<$ (non-numeric).
- *Sometimes* the bits are naturally grouped, e.g. as characters in a string or hexadecimal digits in a number (4 bits), or in general bytes (8 bits).
- Today's sorting algorithms are allowed access these bits or groups of bits, instead of just letting them compare keys . . .
  This was NOT possible in comparison-based setting.

Easy results . . . Surprising results

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Surprising case: (I think)
For any constant k, the problem of sorting n integers in the range $\{1, \ldots, n^k\}$ can be done in $\Theta(n)$ time.

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2. The counting information stored in C can be used to determine the position of each element in the sorted array. Suppose we modify the values of the $C[j]$ so that *now*

   $C[j] =$ the number of keys *less than or equal* to j.

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3. We use a "trick" to move the elements to the right position of an auxiliary array. Then we copy the sorted auxiliary array back to the original one.

Implementation of Counting Sort

Algorithm Counting Sort(A, m)

1. $n \leftarrow A$.length
2. Initialise array $C[1 \dots m]$
3. **for** $i \leftarrow 1$ **to** n **do**
4. $j \leftarrow A[i]$.key
5. $C[j] \leftarrow C[j] + 1$
6. **for** $j \leftarrow 2$ **to** m **do**
7. $C[j] \leftarrow C[j] + C[j-1]$ $\triangleright$ $C[j]$ stores $\sharp$ of keys $\leq j$
8. Initialise array $B[1 \dots n]$
9. **for** $i \leftarrow n$ **downto** 1 **do**
10. $j \leftarrow A[i]$.key $\triangleright$ $A[i]$ highest w. key j
11. $B[C[j]] \leftarrow A[i]$ $\triangleright$ Insert $A[i]$ into highest free index for j
12. $C[j] \leftarrow C[j] - 1$
13. **for** $i \leftarrow 1$ **to** n **do**
14. $A[i] \leftarrow B[i]$

Analysis of Counting Sort

- The loops in lines 3–5, 9–12, and 13–14 all require time $\Theta(n)$.
- The loop in lines 6–7 requires time $\Theta(m)$.
- Thus the overall running time is

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Note: COUNTING-SORT is STABLE.

- (*After* sorting, 2 items with the same key have their *initial relative order*).

Radix Sort

Basic Assumption

Keys are sequences of digits in a fixed range $0, \ldots, R-1$, all of equal length d.

Examples of such keys

- 4 digit hexadecimal numbers (corresponding to 16 bit integers)
  $R = 16, d = 4$
- 5 digit decimal numbers (for example, US post codes)
  $R = 10, d = 5$
- Fixed length ASCII character sequences
  $R = 128$
- Fixed length byte sequences
  $R = 256$

Stable Sorting Algorithms

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Examples

- COUNTING-SORT, MERGE-SORT, and INSERTION SORT are all stable.
- QUICKSORT is not stable.

Radix Sort (cont'd)

Idea

Sort the keys digit by digit, *starting with the least significant digit.*

Example

now		sob		tag		ace
for		nob		ace		bet
tip		ace		bet		dim
ilk		tag		dim		for
dim		ilk		tip		hut
tag		dim		sky		ilk
jot	$\longrightarrow$	tip	$\longrightarrow$	ilk	$\longrightarrow$	jot
sob		for		sob		nob
nob		jot		nob		now
sky		hut		for		sky
hut		bet		jot		sob
ace		now		now		tag
bet		sky		hut		tip

Each of the three sorts is carried out with respect to the digits in that column. "Stability" (and having previously sorted digits/suffixes to the right), means this achieves a sorting of the suffixes starting at the current column.

Radix Sort (cont'd)

Algorithm RADIX-SORT(A, d)

1. **for** $i \leftarrow 0$ **to** d **do**
2.   use stable sort to sort array A using digit i as key

Most commonly, COUNTING SORT is used in line 2 - this means that once a set of digits is already in sorted order, then (by stability) performing COUNTING SORT on the *next-most significant* digits preserves that order, within the "blocks" constructed by the new iteration.

Then each execution of line 2 requires time $\Theta(n + R)$.
Thus the overall time required by RADIX-SORT is

$$\Theta(d(n + R))$$

Sorting Integers with Radix-Sort

Theorem 2

An array of length n whose keys are b-bit numbers can be sorted in time

$$\Theta(n\lceil b/\lg n\rceil)$$

using a suitable version of RADIX-SORT.

Proof: Let the digits be blocks of $\lceil \lg n\rceil$ bits. Then $R = 2^{\lceil \lg n\rceil} = \Theta(n)$ and $d = \lceil b/\lceil \lg n\rceil\rceil$. Using the implementation of RADIX-SORT based on COUNTING SORT the integers can be sorted in time

$$\Theta(d(n + R)) = \Theta(n\lceil b/\lg n\rceil).$$

Note: If all numbers are at most n^k, then $b = k \lg n \ldots \Rightarrow$ Radix Sort is $\Theta(n)$ (assuming k is some constant, e.g., 3, 10).

Reading Assignment

[CLRS] Sections 8.2, 8.3

Problems

1. Think about the qn. on slide 7 - how do we get a very easy (non-stable) version of COUNTING-SORT if there are no items attached to the keys?
2. Can you come up with another way of achieving counting sort's $O(m+n)$-time bound and stability (you will need a different data structure from an array).
3. Exercise 8.3-4 of [CLRS].