Algorithms and Data Structures: Counting sort and Radix sort

Special Cases of the Sorting Problem
In this lecture we assume that the sort keys are sequences of bits.

- Quite a natural special case. Doesn’t cover everything:
  - e.g., exact real number arithmetic doesn’t take this form.
  - In certain applications, e.g. in Biology, pairwise experiments may only return \( > \) or \( < \) (non-numeric).
- Sometimes the bits are naturally grouped, e.g. as characters in a string or hexadecimal digits in a number (4 bits), or in general bytes (8 bits).
- Today’s sorting algorithms are allowed access these bits or groups of bits, instead of just letting them compare keys . . .
  This was NOT possible in comparison-based setting.

Easy results ... Surprising results

Simplest Case:
Keys are integers in the range \( 1, \ldots, m \), where \( m = O(n) \) (\( n \) is (as usual)
the number of elements to be sorted). We can sort in \( \Theta(n) \) time
Simplest Case:
Keys are integers in the range 1, ..., m, where m = O(n) (n is (as usual) the number of elements to be sorted). We can sort in Θ(n) time (big deal ... but will help later).

Surprising case: (I think)
For any constant k, the problem of sorting n integers in the range {1, ..., nk} can be done in Θ(n) time.
Counting Sort

Assumption: Keys (attached to items) are Integers in range 1,\ldots, m.

Idea

1. Count for every key \( j \), \( 1 \leq j \leq m \) how often it occurs in the input array. Store results in an array \( C \).
2. The counting information stored in \( C \) can be used to determine the position of each element in the sorted array. Suppose we modify the values of the \( C[j] \) so that now
   \[ C[j] = \text{the number of keys less than or equal to } j. \]
   Then we know that the elements with key “\( j \)” must be stored at the indices \( C[j − 1] + 1, \ldots, C[j] \) of the final sorted array.

Algorithm Counting Sort\( (A, m) \)

1. \( n \leftarrow A.\text{length} \)
2. Initialise array \( C[1 \ldots m] \)
3. for \( i \leftarrow 1 \) to \( n \) do
   4. \( j \leftarrow A[i].\text{key} \)
   5. \( C[j] \leftarrow C[j] + 1 \)
4. for \( j \leftarrow 2 \) to \( m \) do
5. \( C[j] \leftarrow C[j] + C[j − 1] \quad \triangleright \text{ } C[j] \text{ stores } \sharp \text{ of keys } \leq j \)
6. Initialise array \( B[1 \ldots n] \)
7. for \( i \leftarrow n \) downto 1 do
8. \( j \leftarrow A[i].\text{key} \quad \triangleright \text{ } A[i] \text{ highest w. key } j \)
9. \( B[C[j]] \leftarrow A[i] \quad \triangleright \text{ } \text{Insert } A[i] \text{ into highest free index for } j \)
10. \( C[j] \leftarrow C[j] − 1 \)
11. for \( i \leftarrow 1 \) to \( n \) do
12. \( A[i] \leftarrow B[i] \)

Counting Sort

Assumption: Keys (attached to items) are Integers in range 1,\ldots, m.

Implementation of Counting Sort

Analysis of Counting Sort

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Analysis of Counting Sort

- The loops in lines 3–5, 9–12, and 13–14 all require time $\Theta(n)$.
- The loop in lines 6–7 requires time $\Theta(m)$.
- Thus the overall running time is $O(n + m)$.
- This is linear in the number of elements if $m = O(n)$.

Note: This does not contradict Theorem 3 from Lecture 7 - that's a result about the general case, where keys have an arbitrary size (and need not even be numeric).

Radix Sort

Basic Assumption

Keys are sequences of digits in a fixed range $0, \ldots, R - 1$, all of equal length $d$.

Examples of such keys
- 4 digit hexadecimal numbers (corresponding to 16 bit integers)
  $R = 16, d = 4$
- 5 digit decimal numbers (for example, US post codes)
  $R = 10, d = 5$
- Fixed length ASCII character sequences
  $R = 128$
- Fixed length byte sequences
  $R = 256$

Stable Sorting Algorithms

Definition 1

A sorting algorithm is stable if it always leaves elements with equal keys in their original order.
Stable Sorting Algorithms

**Definition 1**
A sorting algorithm is **stable** if it always leaves elements with equal keys in their original order.

**Examples**
- **Counting-Sort**, **Merge-Sort**, and **Insertion Sort** are all stable.
- **Quicksort** is not stable.

Radix Sort (cont’d)

**Idea**
Sort the keys digit by digit, starting with the least significant digit.

**Example**

```
now  sob  tag  ace
for  nob  ace  bet
tip  ace  bet  dim
dim  ilk  tag  for
tag  dim  sky  ilk
jot  tip  ilk  jot
sob  for  sob  nob
nob  jot  nob  now
sky  hut  for  sky
hut  bet  jot  sob
ace  now  now  tag
bet  sky  hut  tip
```

Each of the three sorts is carried out with respect to the digits in that column. “Stability” (and having previously sorted digits/suffixes to the right), means this achieves a sorting of the suffixes starting at the current column.

**Algorithm** **Radix-Sort** \((A, d)\)

1. for \(i \leftarrow 0\) to \(d\) do
2. use stable sort to sort array \(A\) using digit \(i\) as key

Most commonly, **Counting Sort** is used in line 2 - this means that once a set of digits is already in sorted order, then (by stability) performing **Counting Sort** on the next-most significant digits preserves that order, within the “blocks” constructed by the new iteration.

Then each execution of line 2 requires time \(\Theta(n + R)\).

Thus the overall time required by **Radix-Sort** is

\[ \Theta(d(n + R)) \]

**Theorem 2**
An array of length \(n\) whose keys are \(b\)-bit numbers can be sorted in time

\[ \Theta(n\lceil b/\lg n\rceil) \]

using a suitable version of **Radix-Sort**.

**Proof:** Let the digits be blocks of \(\lceil \lg n \rceil\) bits. Then \(R = 2^{\lceil \lg n \rceil} = \Theta(n)\) and \(d = \lceil b/\lg n \rceil\). Using the implementation of **Radix-Sort** based on **Counting Sort** the integers can be sorted in time

\[ \Theta(d(n + R)) = \Theta(n\lceil b/\lg n\rceil). \]

Note: If all numbers are at most \(n^k\), then \(b = k\lg n \ldots \Rightarrow \) Radix Sort is \(\Theta(n)\) (assuming \(k\) is some constant, e.g., 3, 10).
Reading Assignment

[CLRS] Sections 8.2, 8.3

Problems

1. Think about the qn. on slide 7 - how do we get a very easy (non-stable) version of COUNTING-SORT if there are no items attached to the keys?

2. Can you come up with another way of achieving counting sort’s $O(m + n)$-time bound and stability (you will need a different data structure from an array).

3. Exercise 8.3-4 of [CLRS].

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