Algorithms and Data Structures
Fast Fourier Transform

Complex numbers

Any polynomial $p(x)$ of of degree d ought to have d roots. (I.e., $p(x) = 0$ should have d solutions.)

But the equation

$$x^2 + 1 = 0 \tag{$*$}$$

has no solutions at all if we restrict our attention to real numbers.

Introduce a special symbol i to stand for a solution to $(*)$. Then $i^2 = -1$ and $(*)$ has the required two solutions, i and $-i$.

Adding i allows all polynomial equations to be solved! Indeed a polynomial of degree d has d roots (taking account of multiplicities). This is the *Fundamental Theorem of Algebra*.

Roots of Unity

In particular,

$$x^n = 1$$

has n solutions in the complex numbers. They may be written

$$1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}$$

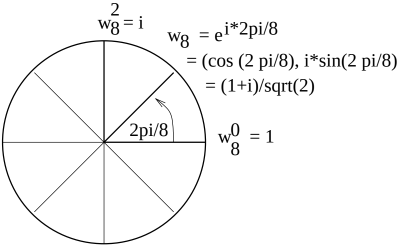
where ω_n is the **principal *n*th root of unity**:

$$\omega_n \;=\; \cos(2\pi/n) + i\sin(2\pi/n), \qquad (\dagger).$$

Convention: from now on ω_n denotes the principal nth root of unity given by $(\dagger)$.

Note: $e^{iu} = \cos u + i \sin u$ so $\omega_n = e^{2\pi i/n}$.

8th Roots of Unity



"Wheel" representation of 8th roots-of-unity (complex plane)).
Same wheel structure for any n (then ω_n found at angle $2\pi/n$).

The Discrete Fourier Transform (DFT)

Instance A sequence of n complex numbers

$$a_0, a_1, a_2, \ldots, a_{n-1},$$

n IS A POWER-OF-2.

Output The sequence of n complex numbers

$$A(1), A(\omega_n), A(\omega_n^2), \ldots, A(\omega_n^{n-1})$$

obtained by evaluating the polynomial

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

at the nth roots of unity.

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The DFT is a *fingerprint* of size n of a polynomial.

It is not the only fingerprint. Given n distinct points, one obtains n equations for the n unknown coefficients of a polynomial of degree $n-1$.

Motivation for algorithms for DFT/Inverse DFT

Direct. Signal processing: mapping between time and frequency domains.

Indirect. Subroutine in numerous applications, e.g., multiplying polynomials or large integers, cyclic string matching, etc.

It is important, therefore to find the fastest method. There is an obvious $\Theta(n^2)$ algorithm. Can we do better?

YES! Really cool algorithm (Fast Fourier Transform (FFT)) runs in $O(n \lg n)$ time. Published by Cooley & Tukey in 1965 - basics known by Gauss in 1805!

Used in *every* Digital Signal Processing application. Probably the most Important algorithm of today. We will show how to apply FFT to do polynomial multiplication in $O(n \lg n)$ (not most common application, but cute).

Divide-and-Conquer

We are interested in evaluating:

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1},$$

n A POWER-OF-2. Put

$$\begin{aligned} A_{\mathrm{even}}(y) &= a_0 + a_2 y + \cdots + a_{n-2} y^{n/2-1}, \\ A_{\mathrm{odd}}(y) &= a_1 + a_3 y + \cdots + a_{n-1} y^{n/2-1}, \end{aligned}$$

so that

$$A(x) = A_{\mathrm{even}}(x^2) + x\, A_{\mathrm{odd}}(x^2). \qquad (\#)$$

To evaluate $A(x)$ at the nth roots of unity, we need to evaluate $A_{\mathrm{even}}(y)$ and $A_{\mathrm{odd}}(y)$ at the points $1, \omega_n^2, \omega_n^4, \ldots, \omega_n^{2(n-1)}$.

We'll show now that these are DFTs. (wrt $n/2$)

Key Facts

Assuming n is even:

- $\omega_n^2 = (e^{\frac{2\pi i}{n}})^2 = e^{\frac{2\pi i}{n/2}} = \omega_{n/2}$, and
- $\omega_n^{n/2} = (e^{\frac{2\pi i}{n}})^{n/2} = e^{\pi i} = -1$.

Thus we have the following relationships between ω_n and $\omega_{n/2}$:

$$
\begin{array}{cccccccc}
1 & \omega_n^2 & \dots & \omega_n^{n-2} & \omega_n^n & \omega_n^{n+2} & \dots & \omega_n^{2(n-1)} \\
\| & \| & \dots & \| & \| & \| & \dots & \| \\
1 & \omega_{n/2} & \dots & \omega_{n/2}^{n/2-1} & 1 & \omega_{n/2} & \dots & \omega_{n/2}^{n/2-1}
\end{array}
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1 & \omega_{n/2} & \dots & \omega_{n/2}^{n/2-1} & 1 & \omega_{n/2} & \dots & \omega_{n/2}^{n/2-1}
\end{array}$$

So evaluating $A_{\mathrm{odd}}(x), A_{\mathrm{even}}(x)$ at ω^2 for all nth-roots-of-unity (in order to implement (#)), is TWO "sweeps" of evaluating $A_{\mathrm{odd}}(x), A_{\mathrm{even}}(x)$ at the $n/2$th-roots.

"Divide": a warning

In performing the "Divide" part of Divide-and-Conquer to DFT, it was important that the "Divide" was based on **odd/even**.

Suppose we had instead partitioned $A(x)$ into small/larger terms:

$$\begin{aligned} A_{\text{small}}(y) &= a_0 + a_1 y + \cdots + a_{n/2-1} y^{n/2-1}, \\ A_{\text{big}}(y) &= a_{n/2} + a_{n/2+1} y + \cdots + a_{n-1} y^{n/2-1} \end{aligned}$$

Then we would have

$$A(x) = A_{\text{small}}(x) + x^{n/2} A_{\text{big}}(x).$$

However, to evaluate $A(x)$ at the nth roots of unity, we would need to evaluate $A_{\text{small}}(y)$ and $A_{\text{big}}(y)$ at all of the nth roots of unity.

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However, to evaluate $A(x)$ at the nth roots of unity, we would need to evaluate $A_{\mathrm{small}}(y)$ and $A_{\mathrm{big}}(y)$ at all of the nth roots of unity.

So for recursive calls: we would reduce the degree of the polynomial (to $n/2 - 1$), but would NOT reduce the "number of roots". We would lose the relationship between degree of poly. and number of roots, which is CRUCIAL.

Key Facts (cont'd)

$$
\begin{aligned}
A(1) &= A_{\mathrm{even}}(1) + 1 \cdot A_{\mathrm{odd}}(1) \\[1em]
A(\omega_n) &= A_{\mathrm{even}}(\omega_n^2) + \omega_n\, A_{\mathrm{odd}}(\omega_n^2) \\
&= A_{\mathrm{even}}(\omega_{n/2}) + \omega_n\, A_{\mathrm{odd}}(\omega_{n/2}) \\[1em]
A(\omega_n^2) &= A_{\mathrm{even}}(\omega_{n/2}^2) + \omega_n^2\, A_{\mathrm{odd}}(\omega_{n/2}^2) \\[1em]
&\vdots \\
A(\omega_n^{n/2-1}) &= A_{\mathrm{even}}(\omega_{n/2}^{n/2-1}) + \omega_n^{n/2-1}\, A_{\mathrm{odd}}(\omega_{n/2}^{n/2-1})
\end{aligned}
$$

The x co-efficient on $xA_{\mathrm{odd}}(x^2)$ of $(\#)$ stays positive until $x = \omega_n^{n/2}$.

Key Facts (cont'd)

$$A(\omega_n^{n/2}) = A_{\mathrm{even}}(1) - 1 \cdot A_{\mathrm{odd}}(1)$$

$$A(\omega_n^{n/2+1}) = A_{\mathrm{even}}(\omega_{n/2}) - \omega_n \, A_{\mathrm{odd}}(\omega_{n/2})$$

$$\vdots$$

$$A(\omega_n^{n-1}) = A_{\mathrm{even}}(\omega_{n/2}^{n/2-1}) - \omega_n^{n/2-1} \, A_{\mathrm{odd}}(\omega_{n/2}^{n/2-1})$$

From $\omega_n^{n/2}$ on, the x co-efficient of $xA_{\mathrm{odd}}(x^2)$ of $(\#)$ is negative.
We will use this negative relationship (with the $j < n/2$ case) on lines 8., 9. of our pseudocode.

The Fast Fourier Transform (FFT)

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1},$$

assume n is a power of 2. Compute

$$A(1), A(\omega_n), A(\omega_n^2), \ldots, A(\omega_n^{n-1}), \qquad (*)$$

as follows:

1. If $n = 1$ then $A(x)$ is a constant so task is trivial. Otherwise split A into A_{even} and A_{odd}.
2. By making two recursive calls compute the values of $A_{\mathrm{even}}(y)$ and $A_{\mathrm{odd}}(y)$ at the $(n/2)$ points $1, \omega_{n/2}, \omega_{n/2}^2, \ldots, \omega_{n/2}^{n/2-1}$.
3. Compute the values $(*)$ by using the equation

$$A(x) = A_{\mathrm{even}}(x^2) + x A_{\mathrm{odd}}(x^2).$$

Implementation

Algorithm $\mathrm{FFT}_n(\langle a_0, \ldots, a_{n-1}\rangle)$

if $n = 1$ **then return** $\langle a_0 \rangle$

else

  $\omega_n \leftarrow e^{2\pi i/n}$

  $\omega \leftarrow 1$

  $\langle y_0^{\text{even}}, \ldots, y_{n/2-1}^{\text{even}}\rangle \leftarrow \mathrm{FFT}_{n/2}(\langle a_0, a_2, \ldots, a_{n-2}\rangle)$

  $\langle y_0^{\text{odd}}, \ldots, y_{n/2-1}^{\text{odd}}\rangle \leftarrow \mathrm{FFT}_{n/2}(\langle a_1, a_3, \ldots, a_{n-1}\rangle)$

  for $k \leftarrow 0$ **to** $n/2 - 1$ **do**

    $y_k \leftarrow y_k^{\text{even}} + \omega y_k^{\text{odd}}$

    $y_{k+n/2} \leftarrow y_k^{\text{even}} - \omega y_k^{\text{odd}}$

    $\omega \leftarrow \omega\omega_n$

  return $\langle y_0, \ldots, y_{n-1}\rangle$

Algorithm assumes n is a power of 2 for easy divisibility.
Generally, we can use padding to make n a power of 2.

Analysis

$T(n)$ worst-case running time of FFT.

Lines 1–4: $\Theta(1)$

Lines 5–6: $\Theta(1) + 2T(n/2)$

Loop, 7–10: $\Theta(n)$

Line 11: $\Theta(1)$

Yields the following recurrence:

$$T(n) = 2T(n/2) + \Theta(n).$$

Solution:

$$T(n) = \Theta(n \cdot \lg(n)).$$

The Discrete Fourier Transform

Recall

- The DFT maps a tuple $\langle a_0, \ldots, a_{n-1} \rangle$ to the tuple $\langle y_0, \ldots, y_{n-1} \rangle$ defined by
$$y_j = \sum_{k=0}^{n-1} a_k \omega_n^{jk},$$
where $\omega_n = e^{2\pi i/n}$ is the principal nth root of unity.
- Thus for every n (power of 2) we may view DFT_n as mapping $\mathbb{C}^n \to \mathbb{C}^n$, where $\mathbb{C}$ denote the complex numbers.
- FFT (the Fast Fourier Transform) is an algorithm computing DFT_n in time
$$\Theta(n \lg(n)).$$

The inverse DFT

$$\begin{array}{rcl} \mathrm{DFT}_n : \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ \langle a_0, \ldots, a_{n-1} \rangle & \longmapsto & \langle y_0, \ldots, y_{n-1} \rangle \end{array}$$

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Question

Can we go back from $\langle y_0, \ldots, y_{n-1} \rangle$ to $\langle a_0, \ldots, a_{n-1} \rangle$?

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More precisely:

1. Is DFT_n invertible, that is, is it one-to-one and onto?
2. If the answer to (1) is 'yes', can we compute DFT_n^{-1} efficiently?

An alternative view on the DFT

DFT_n is the linear mapping described by the matrix

$$V_n = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \dots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}.$$

That is, we have

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We will NOT *actually perform* the näive matrix mult.
(we will do *much* better: $O(n \lg n)$)

Inverse of DFT

Claim: V_n is a van-der-Monde matrix and thus invertible.

Proof: Define the following "Inverse" matrix:

$$V_n^{-1} = \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-4} & \dots & \omega_n^{-2(n-1)} \\ 1 & \omega_n^{-3} & \omega_n^{-6} & \dots & \omega_n^{-3(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix}.$$

Inverse of DFT (proof)

Verification: We must check that $V_n V_n^{-1} = I_n$:
Want $\ell\ell$-th entry $= 1$ $\forall \ell$, and ℓj-th entry $= 0$ $\forall \ell, j$ with $\ell \neq j$.
Expanding ...

$$(V_n V_n^{-1})_{\ell j} = \frac{1}{n} \sum_{k=0}^{n-1} \omega_n^{\ell k} \omega_n^{-kj}$$

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&= \begin{cases} 1 & \text{if } \ell = j \text{ (because } \omega_n^{\ell - j} = 1) \\ 0 & \text{otherwise} \end{cases}
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\end{aligned}$$

$(V_n V_n^{-1})_{\ell j} = 0$ case uses the fact that for all $r \neq 0$ $(r = (\ell - j))$

$$\text{we } \textit{have } \sum_{k=0}^{n-1} \omega_n^{rk} = 0.$$

Inverse of DFT

We have shown DFT_n is invertible with

$$\mathrm{DFT}_n^{-1} : \begin{pmatrix} y_0 \\ \vdots \\ y_{n-1} \end{pmatrix} \mapsto V_n^{-1} \begin{pmatrix} y_0 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix}.$$

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Problem

If we are were to apply $V_n^{-1}\langle y_0, \ldots, y_{n-1}\rangle$ directly in order to recover $\langle a_0, \ldots, a_{n-1}\rangle$, the evaluation of $V_n^{-1}\langle y_0, \ldots, y_{n-1}\rangle$ would take $\Theta(n^2)$ time!!!

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Solution

Take another look back at the V_n^{-1} matrix, and see that it is *more-or-less* a "flipped-over" DFT.

Inverse DFT (efficient) Algorithm

ω_n^{-1} is an nth root of unity (though not the principal one). Note that

$$(\omega_n^{-1})^j = 1/\omega_n^j = \omega_n^n/\omega_n^j = \omega_n^{n-j},$$

for every $0 \leq j < n$.

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Inverse FFT

- Compute $\mathrm{DFT}_n\langle y_0, \ldots, y_{n-1}\rangle$ (*deliberately* using DFT_n, not inverse), to obtain the result $\langle d_0, \ldots, d_{n-1}\rangle$.
- Flip the sequence $d_1, d_2, \ldots, d_{n-1}$ in this result (keeping d_0 fixed), then divide every term by n.

$$a_i = \begin{cases} \frac{d_0}{n} & \text{if } i = 0 \\ \frac{d_{n-i}}{n} & \text{if } 1 \leq i \leq n-1 \end{cases}$$

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Worst-case running time is $\Theta(n \lg(n))$.

Our Application! Multiplication of Polynomials

Input:
$$\begin{aligned} p(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} \\ q(x) &= b_0 + b_1 x + b_2 x^2 + \cdots + b_{m-1} x^{m-1}. \end{aligned}$$

Required output:

$$\begin{aligned} p(x)q(x) = \;& (a_0 b_0) \\ & + (a_0 b_1 + a_1 b_0) x \\ & + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 \\ & \quad \vdots \\ & + (a_{n-2} b_{m-1} + a_{n-1} b_{m-2}) x^{n+m-3} \\ & + (a_{n-1} b_{m-1}) x^{n+m-2} \end{aligned}$$

Naive method uses $\Theta(nm)$ arithmetic operations

CAN WE DO BETTER?

Interpolation

Theorem

Let $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{C}$ *pairwise distinct and* $y_0, \ldots, y_{n-1} \in \mathbb{C}$*.*
Then there exists exactly one polynomial $p(X)$ *of degree at most* $n-1$ *such that for* $0 \leq k \leq n-1$

$$p(\alpha_k) = y_k.$$

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$$p(\alpha_k) = y_k.$$

- The sequence

$$\langle (\alpha_0, y_0), \ldots, (\alpha_{n-1}, y_{n-1}) \rangle$$

  is called a point-value representation of the polynomial p.
- The process of computing a polynomial from a point-value representation is called **interpolation**.

Multiplication of polynomials (cont'd)

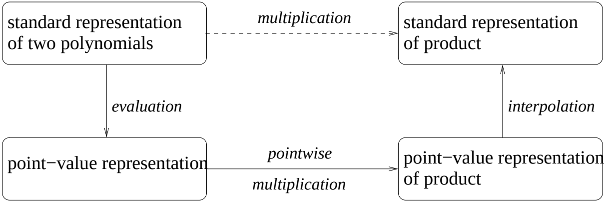
Observation

Suppose we have two polynomials $p(X)$ (of degree $n-1$) and $q(X)$ (of degree $m-1$). Assume $\max\{m,n\}=n$. If $\langle(\alpha_0,y_0),\ldots,(\alpha_{n+m-2},y_{n+m-2})\rangle$ and $\langle(\alpha_0,z_0),\ldots,(\alpha_{n+m-2},z_{n+m-2})\rangle$ are point-value representations $p(X)$ and $q(X)$ respectively (evaluated at exactly the same points), then

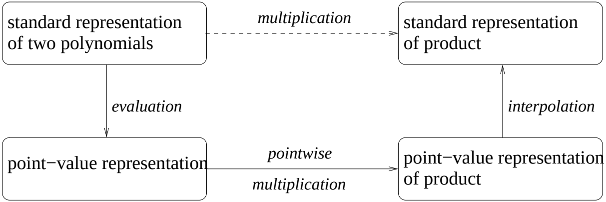
$$\langle(\alpha_0,y_0z_0),\ldots,(\alpha_{n+m-2},y_{n+m-2}z_{n+m-2})\rangle$$

is a point-value representation of $p(X)q(X)$ (with enough points to allow us to recover $pq(X)$ by interpolation) .

Multiplication of polynomials (cont'd)



Multiplication of polynomials (cont'd)



we take the solid-arrow route, using 3 steps, to achieve performance $\Theta(n \lg(n))$.

Multiplication of polynomials (cont'd)

Key idea

Let n' be the smallest power of 2 such that $n' \geq n + m - 1$.
Use the n'-th roots of unity as the evaluation points:
$\alpha_0 = 1$, $\alpha_1 = \omega_{n'}$, $\alpha_2 = \omega_{n'}^2$, $\ldots$, $\alpha_{n'-1} = \omega_{n'}^{n'-1}$.
Then

- evaluation $\equiv$ DFT, and
- interpolation $\equiv$ inverse DFT

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Overall running time is

$$
\begin{array}{rll}
 & \Theta(n' \log n') = \Theta(n \log n) & \text{(FFT)} \\
+ & \Theta(n') = \Theta(n) & \text{(pointwise multiplication)} \\
+ & \Theta(n' \log n') = \Theta(n \log n) & \text{(inverse FFT)} \\
\hline
= & \Theta(n \log n) &
\end{array}
$$

Reading Assignment

[CLRS] (2nd and 3rd ed) Section 30.2 and 30.3.

Problems

1. Exercise 30.2-2 of [CLRS].
2. Let $f(x) = 3\cos(2x)$. For $0 \leq k \leq 3$, let $a_k = f(2\pi k/4)$. Compute the DFT of $\langle a_0, \ldots, a_3 \rangle$.
   Do the same for $f(x) = 5\sin(x)$.
3. Exercise 30.2-3 of [CLRS].
4. Exercise 30.2-7 of [CLRS].