# Algorithms and Data Structures
# Strassen's Algorithm

# Tutorials

- Start in week (week 3)
- Tutorial allocations are linked from the course webpage
  `http://www.inf.ed.ac.uk/teaching/courses/ads/`

# The Master Theorem for solving recurrences

## Theorem

*Let $n_0 \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $a, b \in \mathbb{R}$ with $a > 0$ and $b > 1$, and let $T : \mathbb{N} \to \mathbb{R}$ satisfy the following recurrence:*

$$T(n) = \begin{cases} \Theta(1) & \text{if } n < n_0, \\ a \cdot T(n/b) + \Theta(n^k) & \text{if } n \geq n_0. \end{cases}$$

*Let $c = \log_b(a)$; we call $c$ the critical exponent. Then*

$$T(n) = \begin{cases} \Theta(n^c) & \text{if } k < c & \text{(I)}, \\ \Theta(n^c \cdot \lg(n)) & \text{if } k = c & \text{(II)}, \\ \Theta(n^k) & \text{if } k > c & \text{(III)}. \end{cases}$$

*Theorem also holds if we replace $a \cdot T(n/b)$ above by $a_1 \cdot T(\lfloor n/b \rfloor) + a_2 \cdot T(\lceil n/b \rceil)$ for any $a_1, a_2 \geq 0$ with $a_1 + a_2 = a$.*

# The Master Theorem (cont'd)

- We don't have time to prove the Master Theorem in class. You can find the proof in Section 4.6 of [CLRS]. *Section 4.4 of [CLRS], 2nd ed*.
  Their version of the M.T. is a bit more general than ours.
- Consider the following examples:

$$\begin{aligned} T(n) &= 4T(n/2) + n, \\ T(n) &= 4T(\lfloor n/2 \rfloor) + n^2, \\ T(n) &= 4T(n/2) + n^3. \end{aligned}$$

  Could alternatively unfold-and-sum to prove the first and third of these (and to get an estimate for the second).

CLASS EXERCISE

# Matrix Multiplication

## Recall

The product of two $(n \times n)$-matrices

$$A = (a_{ij})_{1 \leq i,j \leq n} \quad \text{and} \quad B = (b_{ij})_{1 \leq i,j \leq n}$$

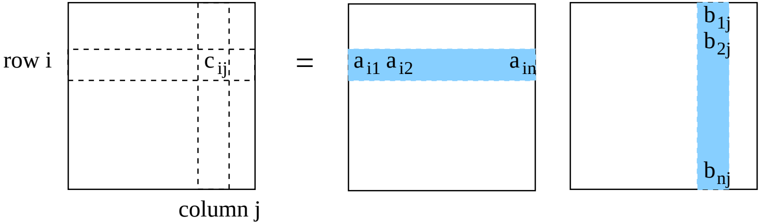
is the $(n \times n)$-matrix $C = AB$ where $C = (c_{ij})_{1 \leq i,j \leq n}$ with entries

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

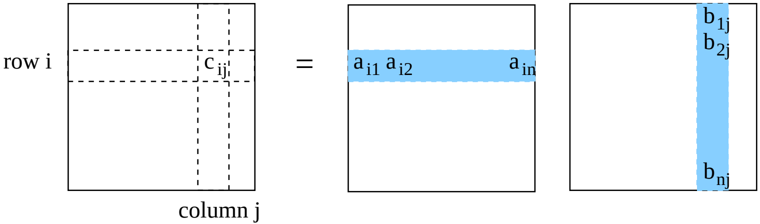
## The Matrix Multiplication Problem

*Input:* $(n \times n)$-matrices $A$ and $B$
*Output:* the $(n \times n)$-matrix $AB$

# Matrix Multiplication



# Matrix Multiplication




- $n$ multiplications and $n$ additions for each $c_{ij}$.
- there are $n^2$ different $c_{ij}$ entries.

# A straightforward algorithm

**Algorithm** $\textsc{MatMult}(A, B)$

$n \leftarrow$ number of rows of $A$
**for** $i \leftarrow 1$ **to** $n$ **do**
    **for** $j \leftarrow 1$ **to** $n$ **do**
        $c_{ij} \leftarrow 0$
        **for** $k \leftarrow 1$ **to** $n$ **do**
            $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$
**return** $C = (c_{ij})_{1 \leq i,j \leq n}$

Requires

$$\Theta(n^3)$$

arithmetic operations (additions and multiplications).

# A näive divide-and-conquer algorithm

Observe

If

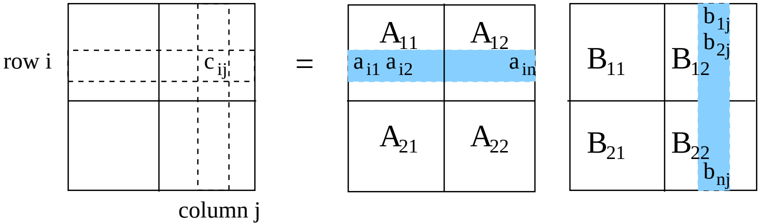
$$A = \left( \begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right) \quad \text{and} \quad B = \left( \begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array} \right)$$

for $(n/2 \times n/2)$-submatrices $A_{ij}$ and $B_{ij}$ then

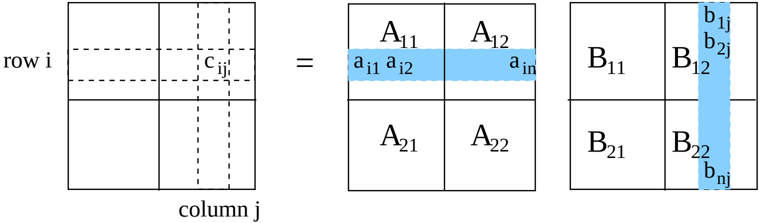
$$AB = \left( \begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array} \right)$$

note: We are assuming $n$ is a power of 2.

# A näive divide-and-conquer algorithm



# A näive divide-and-conquer algorithm




Suppose $i \leq n/2$ and $j > n/2$. Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \underbrace{\sum_{k=1}^{n/2} a_{ik} b_{kj}}_{\in A_{11} B_{12}} + \underbrace{\sum_{k=n/2+1}^{n} a_{ik} b_{kj}}_{\in A_{12} B_{22}}$$

# A näive divide-and-conquer algorithm (cont'd)

*Assume $n$ is a power of 2.*

**Algorithm** D&C-MatMult($A, B$)

1. $n \leftarrow$ number of rows of $A$
2. **if** $n = 1$ **then return** $(a_{11} b_{11})$
3. **else**
4. Let $A_{ij}$, $B_{ij}$ (for $i, j = 1, 2$) be $(n/2 \times n/2)$-submatrices s.th.
   $$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \text{ and } B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$$
5. Recursively compute $A_{11}B_{11}$, $A_{12}B_{21}$, $A_{11}B_{12}$, $A_{12}B_{22}$, $A_{21}B_{11}$, $A_{22}B_{21}$, $A_{21}B_{12}$, $A_{22}B_{22}$
6. Compute $C_{11} = A_{11}B_{11} + A_{12}B_{21}$, $C_{12} = A_{11}B_{12} + A_{12}B_{22}$, $C_{21} = A_{21}B_{11} + A_{22}B_{21}$, $C_{22} = A_{21}B_{12} + A_{22}B_{22}$
7. **return** $\left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right)$

# Analysis of D&C-MatMult

$T(n)$ is the number of operations done by D&C-MatMult.

- Lines $1, 2, 3, 4, 7$ require $\Theta(1)$ arithmetic operations
- Line 5 requires $8T(n/2)$ arithmetic operations
- Line 6 requires $4(n/2)^2 = \Theta(n^2)$ arithmetic operations.
  **Remember!** Size of matrices is $\Theta(n^2)$, NOT $\Theta(n)$

We get the recurrence

$$T(n) = 8T(n/2) + \Theta(n^2).$$

Since $\log_2(8) = 3$, the Master Theorem yields

$$T(n) = \Theta(n^3).$$

# Analysis of D&C-MatMult

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(No improvement over MatMult ... why? CLASS? ...)

# Strassen's algorithm (1969)

*Assume* $n$ *is a power of* 2.

Let

$$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \quad \text{and} \quad B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right).$$

We want to compute

$$\begin{aligned} AB &= \left(\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right) \\ &= \left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right). \end{aligned}$$

Strassen's algorithm uses a *trick* in applying Divide-and-Conquer.

# Strassen's algorithm (cont'd)

Let

$$
\begin{aligned}
P_1 &= (A_{11} + A_{22})(B_{11} + B_{22}) \\
P_2 &= (A_{21} + A_{22})B_{11} \\
P_3 &= A_{11}(B_{12} - B_{22}) \\
P_4 &= A_{22}(-B_{11} + B_{21}) \qquad (*) \\
P_5 &= (A_{11} + A_{12})B_{22} \\
P_6 &= (-A_{11} + A_{21})(B_{11} + B_{12}) \\
P_7 &= (A_{12} - A_{22})(B_{21} + B_{22})
\end{aligned}
$$

# Strassen's algorithm (cont'd)

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P_1 &= (A_{11} + A_{22})(B_{11} + B_{22}) \\
P_2 &= (A_{21} + A_{22})B_{11} \\
P_3 &= A_{11}(B_{12} - B_{22}) \\
P_4 &= A_{22}(-B_{11} + B_{21}) \\
P_5 &= (A_{11} + A_{12})B_{22} \\
P_6 &= (-A_{11} + A_{21})(B_{11} + B_{12}) \\
P_7 &= (A_{12} - A_{22})(B_{21} + B_{22})
\end{aligned} \qquad (*)$$

Then

$$\begin{aligned}
C_{11} &= P_1 + P_4 - P_5 + P_7 & C_{12} &= P_3 + P_5 \\
C_{21} &= P_2 + P_4 & C_{22} &= P_1 + P_3 - P_2 + P_6
\end{aligned} \qquad (**)$$

# Checking Strassen's algorithm - $C11$

We will check the equation for $C_{11}$ is correct.
Strassen's algorithm computes $C_{11} = P1 + P4 - P5 + P7$. We have

$$\begin{aligned}
P1 &= (A11 + A22)(B11 + B22) \\
&= A11B11 + A11B22 + A22B11 + A22B22. \\
P4 &= A22(-B11 + B21) \;=\; A22B21 - A22B11. \\
P5 &= (A11 + A12)B22 \;=\; A11B22 + A12B22. \\
P7 &= (A12 - A22)(B21 + B22) \\
&= A12B21 + A12B22 - A22B21 - A22B22.
\end{aligned}$$

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\end{aligned}$$

Then $P1 + P4 = A11B11 + A11B22 + A22B22 + A22B21$.

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Then $P1 + P4 = A11B11 + A11B22 + A22B22 + A22B21$.
Then $P1 + P4 - P5 = A11B11 + A22B22 + A22B21 - A12B22$.

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P7 &= (A12 - A22)(B21 + B22) \\
&= A12B21 + A12B22 - A22B21 - A22B22.
\end{aligned}
$$

Then $P1 + P4 = A11B11 + A11B22 + A22B22 + A22B21$.
Then $P1 + P4 - P5 = A11B11 + A22B22 + A22B21 - A12B22$.
Then $P1 + P4 - P5 + P7 = A11B11 + A12B21$, which is $C11$.

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P7 &= (A12 - A22)(B21 + B22) \\
&= A12B21 + A12B22 - A22B21 - A22B22.
\end{aligned}$$

Then $P1 + P4 = A11B11 + A11B22 + A22B22 + A22B21$.
Then $P1 + P4 - P5 = A11B11 + A22B22 + A22B21 - A12B22$.
Then $P1 + P4 - P5 + P7 = A11B11 + A12B21$, which is $C11$.

**Homework:** check other 3 equations.

# Strassen's algorithm (cont'd)

## Crucial Observation

Only **7** multiplications of $(n/2 \times n/2)$-matrices are needed to compute $AB$.

**Algorithm** $\textsc{Strassen}(A, B)$

1. $n \leftarrow$ number of rows of $A$
2. **if** $n = 1$ **then return** $(a_{11} b_{11})$
3. **else**
4. Determine $A_{ij}$ and $B_{ij}$ for $i, j = 1, 2$ (as before)
5. Compute $P_1, \ldots, P_7$ as in $(*)$
6. Compute $C_{11}, C_{12}, C_{21}, C_{22}$ as in $(**)$
7. **return** $\left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right)$

# Analysis of Strassen's algorithm

Let $T(n)$ be the number of arithmetic operations performed by STRASSEN.

- Lines $1-4$ and 7 require $\Theta(1)$ arithmetic operations
- Line 5 requires $7T(n/2)+\Theta(n^2)$ arithmetic operations
- Line 6 requires $\Theta(n^2)$ arithmetic operations. remember.

We get the recurrence

$$T(n) = 7T(n/2) + \Theta(n^2).$$

Since $\log_2(7) \approx 2.807 > 2$, the Master Theorem yields

$$T(n) = \Theta(n^{\log_2(7)}).$$

# Breakthroughs on matrix multiplication

- Coppersmith & Winograd (1987) came up with an improved algorithm with running time of

$$O(n^{2.376}).$$

- *... many years of silence ...*
- Then in his 2010 PhD thesis, **Andrew Stothers** from the School of Maths, at the **University of Edinburgh** got an algorithm with $O(n^c)$ for $c < 2.3737$ ...
  - $\Rightarrow$ Coppersmith/Winograd not optimal.
  - But Stothers didn't publish.
- In 2011, Virginia Vassilevska Williams of Stanford, came up with a $O(n^c)$ algorithm, for $c = 2.3729$ (partly, but not only, making use of some of Stothers' ideas)
- 2014, François Le Gall, $O(n^c)$ algorithm, for $c = 2.3728639$.

# Remarks on Matrix Multiplication

- In practice, the "school" MATMULT algorithm tends to outperform Strassen's algorithm, unless the matrices are huge.
- The best known lower bound for matrix multiplication is

$$\Omega(n^2).$$

This is a *trivial* lower bound (need to look at all entries of each matrix). Amazingly, $\Omega(n^2)$ is believed to be "the truth"!

Open problem: Can we find a $O(n^{2+o(1)})$-algorithm for Matrix Multiplication of $n \times n$ matrices?

# Reading Assignment

[CLRS] (3rd ed) Section 4.5 “The Master method for solving recurrences”
(*Section 4.3 “Using the Master method” of [CLRS], 2nd ed*)
[CLRS] (3rd ed) Section 4.2 (*Section 28.2 of [CLRS], 2nd ed*)

**Problems**

1. Exercise 4.5-2 of [CLRS] (3rd ed) *Exercise 4.3-2 of [CLRS], 2nd ed.*
2. Exercise 4.2-1 of [CLRS], 3rd ed. *Exercise 28.2-1 [CLRS], 2nd ed.*
3. Week 3 tutorial sheet.