Algorithms and Data Structures: Network Flows

Flow Networks

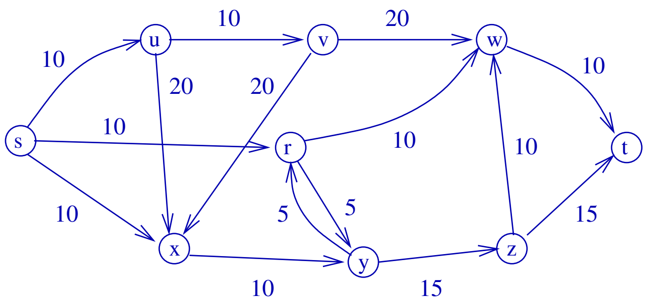
Definition 1
A *flow network* consists of

- A directed graph $\mathcal{G} = (V, E)$.
- A *capacity function* $c : V \times V \to \mathbb{R}$ such that $c(u, v) \geq 0$ if $(u, v) \in E$ and $c(u, v) = 0$ for all $(u, v) \notin E$.
- Two distinguished vertices $s, t \in V$ called the *source* and the *sink*, respectively.

We read (u, v) to mean $u \to v$.

Assumption
Each vertex $v \in V$ is on some *directed path* from s to t. This implies that $\mathcal{G}$ is connected (but not necessarily strongly connected), and that $|E| \geq |V| - 1$.

Example



For this graph, $V = \{s, r, u, v, w, x, y, z, t\}$. The edge set is

$$E = \{(s,u), (s,r), (s,x), (u,v), (u,x), (v,x), (v,w), (r,w), (r,y), (x,y), (y,r), (y,z), (z,w), (z,t), (w,t)\}.$$

Some examples of *capacities* are $c(s,x) = 10$, $c(r,y) = 5$, $c(v,x) = 20$ and $c(v,r) = 0$ (since there is no arc from v to r).

Network Flows

Definition 2
Let $\mathcal{N} = (\mathcal{G} = (V, E), c, s, t)$ be a flow network.
A *flow* in $\mathcal{N}$ is a function

$$f : V \times V \to \mathbb{R}$$

satisfying the following conditions:

Capacity constraint: $f(u, v) \leq c(u, v)$ for all $u, v \in V$.

Skew symmetry: $f(u, v) = -f(v, u)$ for all $u, v \in V$.

Flow conservation: For all $u \in V \setminus \{s, t\}$,

$$\sum_{v \in V} f(u, v) = 0.$$

Network Flows (cont'd)

$\mathcal{N} = (\mathcal{G} = (V, E), c, s, t)$ flow network, $f : V \times V \to \mathbb{R}$ flow in $\mathcal{N}$.

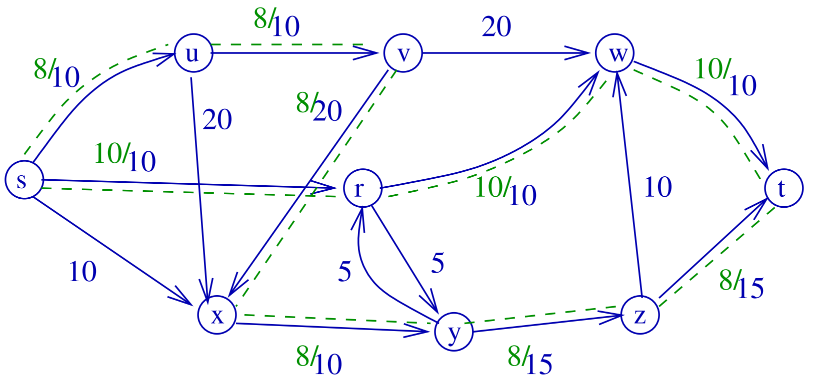
- For $u, v \in V$ we call $f(u, v)$ the *net flow* at (u, v).
- The *value* of the flow f is the number

$$|f| = \sum_{v \in V} f(s, v).$$

Notice that our particular defn. of flow (the "skew-symmetry" constraint) ensures that $f(u, v)$ is truly the "net flow" in the usual sense of the word (e.g. if (r, y) on slide 2 was to carry flow 3, and (y, r) to carry flow 4, we will have $f(r, y) = -1$).

Example

A flow of value 18.



Only positive net flows are shown.

The Maximum-Flow Problem

Input: Network $\mathcal{N}$
Output: Flow of maximum value in $\mathcal{N}$

The problem is to find the flow f such that $|f| = \sum_{v \in V} f(s, v)$ is the largest possible (over all "legal" flows).

The Ford-Fulkerson Algorithm

Published in 1956 by Delbert Fulkerson and Lester Randolph Ford Jr.

Algorithm FORD-FULKERSON($\mathcal{N}$)

1. $f \leftarrow$ flow of value 0
2. **while** there exists an $s \to t$ path $\mathcal{P}$ in the "residual network" **do**
3. $\quad f \leftarrow f + f_{\mathcal{P}}$;
4. $\quad$ Update the "residual network".
5. **return** f

The "residual network" is $\mathcal{N}$ with the "used-up" capacity removed.

To make this precise, we need notation, and proofs - this lecture.

Some Technical Observations

$\mathcal{N} = (\mathcal{G} = (V, E), c, s, t)$ flow network, $f : V \times V \to \mathbb{R}$ flow in $\mathcal{N}$, $u, v \in V$.

1. $f(u, u) = 0$ for all $u \in V$.

   "Proof": $f(u, u) = -f(u, u)$ by skew symmetry.

2. For any $v \in V \setminus \{s, t\}$,
$$\sum_{u \in V} f(u, v) = 0.$$

   Proof: $\sum_{u \in V} f(u, v) = -\sum_{u \in V} f(v, u) = 0$ by skew symmetry and flow conservation.

3. If $(u, v) \notin E$ and $(v, u) \notin E$ then $f(u, v) = f(v, u) = 0$.

   Proof: Either $f(u, v)$ or $f(v, u) \geq 0$ by skew symmetry. Say, $f(u, v) \geq 0$. Then $0 \leq f(u, v) \leq c(u, v) = 0$ by the capacity constraint. So $f(u, v) = 0$. By skew symmetry, this shows $f(v, u) = 0$.

One More Technical Observation

4. The *positive net flow entering* v is:
$$\sum_{\substack{u \in V \\ f(u,v) > 0}} f(u, v).$$

   The *positive net flow leaving* v is defined symmetrically.

   Flow conservation now says:

   "positive net flow in = positive net flow out".

All these observations are just to make it easy for us to talk about flows.

Working with Flows

Implicit summation notation: For $X, Y \subseteq V$ put
$$f(X, Y) = \sum_{u \in X} \sum_{v \in Y} f(u, v) = \sum_{(u,v) \in X \times Y} f(u, v).$$

Abbreviations:

$f(u, Y)$ stands for $f(\{u\}, Y)$ and
$f(X, v)$ stands for $f(X, \{v\})$.

Conservation of flow is now:
$$f(u, V) = 0 \quad \text{for all } u \in V \setminus \{s, t\}.$$

Working with Flows (cont'd)

Lemma 3

$\mathcal{N} = (\mathcal{G} = (V, E), c, s, t)$ flow network, f flow in $\mathcal{N}$. Then for all $X, Y, Z \subseteq V$,

1. $f(X, X) = 0$.
2. $f(X, Y) = -f(Y, X)$.
3. *If $X \cap Y = \emptyset$ then*
$$\begin{aligned} f(X \cup Y, Z) &= f(X, Z) + f(Y, Z), \\ f(Z, X \cup Y) &= f(Z, X) + f(Z, Y). \end{aligned}$$

Lemma "lifts" Network flow properties to *sets*-of-vertices.

1.
$$\begin{aligned} f(X,X) &= \sum_{(u,v)\in X\times X} f(u,v) && \textit{by defn. of } f(X,X)\\ &= \sum_{\{u,v\}\subseteq X} \big(f(u,v)+f(v,u)\big) && \textit{take } (u,v),\ (v,u) \textit{ together}\\ &= 0. && \textit{by skew-symm} \end{aligned}$$

2.
$$\begin{aligned} f(X,Y) &= \sum_{(u,v)\in X\times Y} f(u,v) && \textit{by defn of } f(X,Y)\\ &= \sum_{(u,v)\in X\times Y} -f(v,u) && \textit{by skew-symmetry}\\ &= -\sum_{(v,u)\in Y\times X} f(v,u) && \textit{take } - \textit{ outside the summation}\\ &= -f(Y,X). && \textit{by defn of } f(Y,X) \end{aligned}$$

3.
$$\begin{aligned} f(X\cup Y,Z) &= \sum_{u\in X\cup Y}\sum_{v\in Z} f(u,v)\\ &= \sum_{u\in X}\sum_{v\in Z} f(u,v) + \sum_{u\in Y}\sum_{v\in Z} f(u,v) - \sum_{u\in X\cap Y}\sum_{v\in Z} f(u,v) \end{aligned}$$

(expand sum into X and Y, subtract duplicates in $X\cap Y$)

$$= \sum_{u\in X}\sum_{v\in Z} f(u,v) + \sum_{u\in Y}\sum_{v\in Z} f(u,v)$$

(but $X\cap Y=\emptyset$, so third term disappears)

$$= f(X,Z)+f(Y,Z).$$

Moreover,

$$f(Z,X\cup Y) = -f(X\cup Y,Z) = -\big(f(X,Z)+f(Y,Z)\big) = f(Z,X)+f(Z,Y).$$

Working with Flows (cont'd)

Corollary 4

$\mathcal{N}=(\mathcal{G}=(V,E),c,s,t)$ flow network, f flow in $\mathcal{N}$. Then

$$|f| = f(V,t).$$

Proof:

$$\begin{aligned} |f| &= f(s,V) && \text{(by definition)}\\ &= f(V,V) - f(V\setminus\{s\},V) && \text{(by Lemma 3 (3.))}\\ &= -f(V\setminus\{s\},V) && \text{(by Lemma 3 (1.))}\\ &= f(V,V\setminus\{s\}) && \text{(by Lemma 3 (2.))}\\ &= f(V,t)+f(V,V\setminus\{s,t\}) && \text{(by Lemma 3 (3.))}\\ &= f(V,t)+\textstyle\sum_{v\in V\setminus\{s,t\}} f(V,v) && \text{(by Definition)}\\ &= f(V,t) && \text{(by flow conservation)} \end{aligned}$$

Residual Networks

Idea is to capture possible extra flow given current flow.

Definition 5

$\mathcal{N}=(\mathcal{G}=(V,E),c,s,t)$ flow network, f flow in $\mathcal{N}$.

1. For all $u,v\in V\times V$, the *residual capacity* of (u,v) is

$$c_f(u,v) = c(u,v) - f(u,v).$$

2. The *residual network* of $\mathcal{N}$ induced by f is

$$\mathcal{N}_f((V,E_f),c_f,s,t),$$

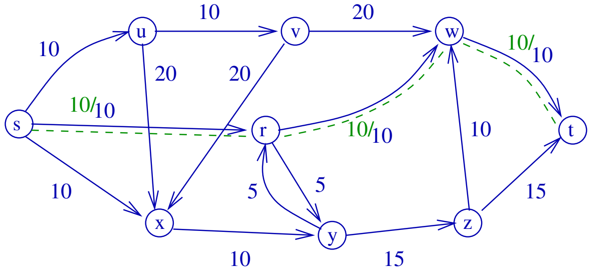
where

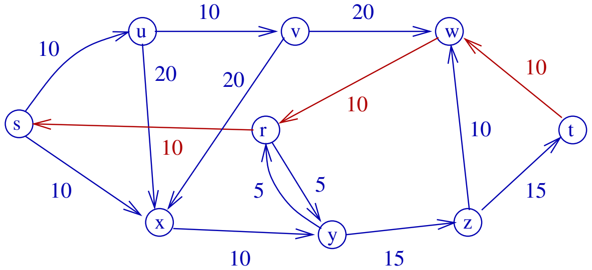
$$E_f = \{(u,v)\in V\times V \mid c_f(u,v)>0\}$$

Notice that E_f may contain edges not originally in E ("back-edges").

Example

A flow and the corresponding residual network





Adding Flows

Lemma 6

Let $\mathcal{N} = (\mathcal{G} = (V, E), c, s, t)$ be a flow network.

Let f be a flow in $\mathcal{N}$.

Let $g : V \times V \to \mathbb{R}$ be a flow in the residual network $\mathcal{N}_f$.

Then the function $f + g : V \times V \to \mathbb{R}$ defined by

$$(f + g)(u, v) = f(u, v) + g(u, v)$$

is a flow of value $|f| + |g|$ in $\mathcal{N}$.

Proof of Lemma 6

First we have to check that $f + g$ is actually a flow in $\mathcal{N}$.

Capacity constraints:

$$\begin{aligned} (f+g)(u,v) &= f(u,v) + g(u,v) \\ &\leq f(u,v) + c_f(u,v) \\ &= f(u,v) + c(u,v) - f(u,v) \\ &= c(u,v). \end{aligned}$$

Skew symmetry:

$$(f+g)(u,v) = f(u,v) + g(u,v) = -f(v,u) - g(v,u) = -(f+g)(v,u).$$

Flow Conservation: For every $u \in V \setminus \{s, t\}$:

$$\sum_{v \in V} (f+g)(u,v) = \sum_{v \in V} f(u,v) + \sum_{v \in V} g(u,v) = 0 + 0 = 0.$$

Proof of Lemma 6 (cont'd)

Next we have to check that $f + g$ does have the value that we claimed for it.

Value:

$$\begin{aligned} |f+g| &= \sum_{v \in V} (f+g)(s,v) \\ &= \sum_{v \in V} f(s,v) + \sum_{v \in V} g(s,v) \\ &= |f| + |g|. \end{aligned}$$

Augmenting Paths

Definition 7

$\mathcal{N} = (\mathcal{G} = (V, E), c, s, t)$ flow network, f flow in $\mathcal{N}$.

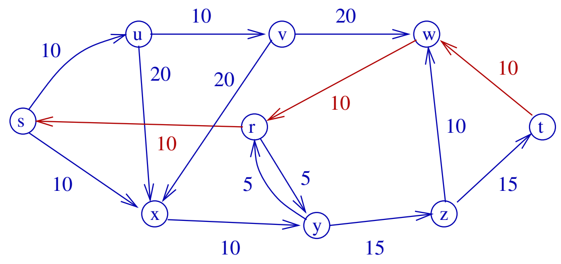
Then an *augmenting path* for f is a path $\mathcal{P}$ from s to t in the residual network $\mathcal{N}_f$.

The *residual capacity* of $\mathcal{P}$ is

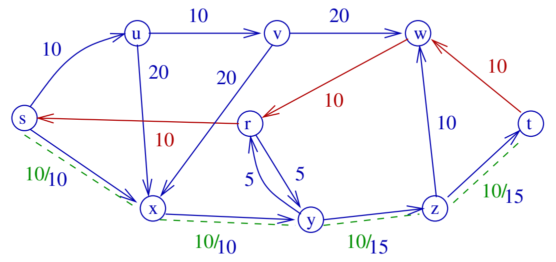
$$c_f(\mathcal{P}) = \min\{c_f(u, v) \mid (u, v) \text{ edge on } \mathcal{P}\}.$$

Note that $c_f(\mathcal{P}) > 0$, by definition of E_f (recall that we only keep edges in E_f if their residual capacity is strictly positive).

Example



An augmenting path of residual capacity 10



Pushing Flow through an Augmenting Path

Lemma 8

$\mathcal{N} = (\mathcal{G} = (V, E), c, s, t)$ flow network, f flow in $\mathcal{N}$. $\mathcal{P}$ augmenting path. Then $f_\mathcal{P} : V \times V \to \mathbb{R}$ defined by

$$f_\mathcal{P}(u, v) = \begin{cases} c_f(\mathcal{P}) & \textit{if } (u, v) \textit{ is an edge of } \mathcal{P}, \\ -c_f(\mathcal{P}) & \textit{if } (v, u) \textit{ is an edge of } \mathcal{P}, \\ 0 & \textit{otherwise} \end{cases}$$

is a flow in $\mathcal{N}_f$ of value $c_f(\mathcal{P})$.

Proof left as an exercise. It is not too difficult - just have to check that the three conditions of a flow are satisfied (and that the value is $c_f(\mathcal{P})$). Similar to Lemma 6.

Augmenting a Flow

Corollary 9

$\mathcal{N} = (\mathcal{G} = (V, E), c, s, t)$ flow network, f flow in $\mathcal{N}$. Let $\mathcal{P}$ be an augmenting path. Then $f + f_\mathcal{P}$ is a flow in $\mathcal{N}$ of value

$$|f| + c_f(\mathcal{P}) > |f|.$$

Proof: Follows from Lemma 6 and Lemma 8.

The Ford-Fulkerson Algorithm

Algorithm FORD-FULKERSON($\mathcal{N}$)

$f \leftarrow$ flow of value 0
while there exists an augmenting path $\mathcal{P}$ in $\mathcal{N}_f$ **do**
    $f \leftarrow f + f_{\mathcal{P}}$
return f

To prove that FORD-FULKERSON correctly solves the Maximum Flow problem, we have to prove that:

1. The algorithm terminates.
2. After termination, f is a maximum flow.

Cuts

Definition 10

$\mathcal{N} = (\mathcal{G} = (V, E), c, s, t)$ flow network.
A *cut* of $\mathcal{N}$ is a pair (S, T) such that:

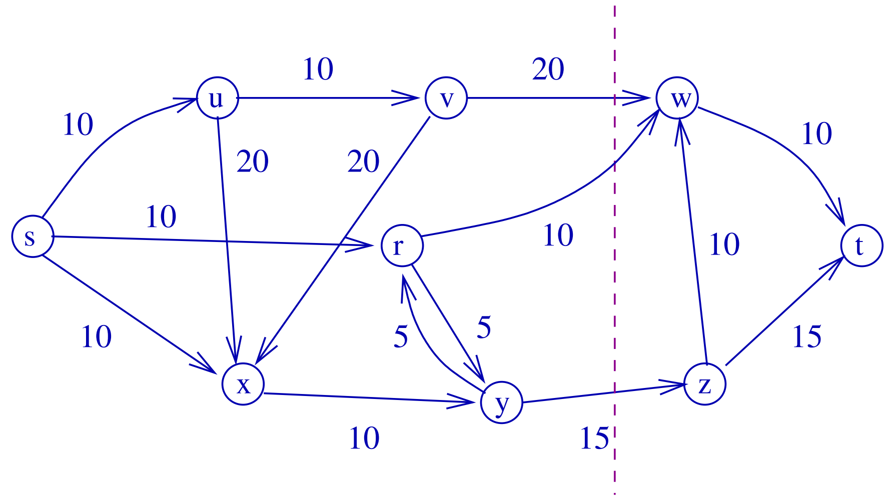
1. $s \in S$ and $t \in T$,
2. $V = S \cup T$ and $S \cap T = \emptyset$.

The *capacity* of the cut (S, T) is

$$c(S, T) = \sum_{u \in S, v \in T} c(u, v).$$

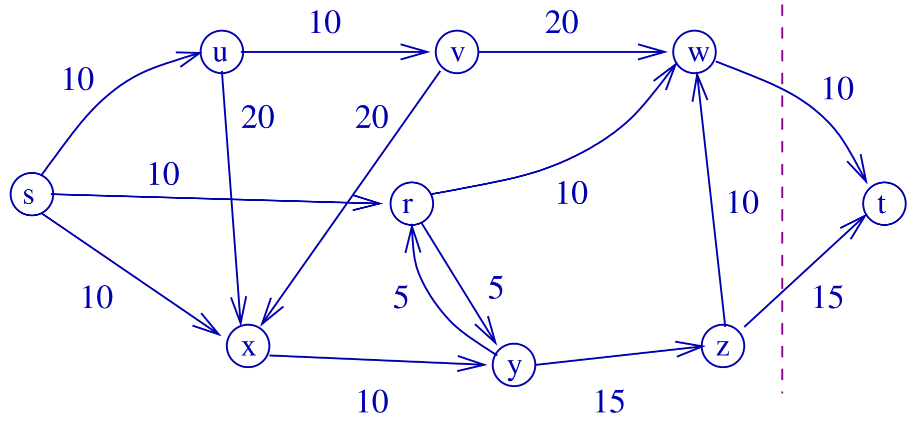
Example

A cut of capacity 45.



Example

A cut of capacity 25.



Cuts and Flows

Lemma 11

$\mathcal{N} = (\mathcal{G} = (V, E), c, s, t)$ flow network, f flow in $\mathcal{N}$, (S, T) cut of $\mathcal{N}$. Then

$$|f| = f(S, T).$$

Proof: We apply Lemma 3:

$$\begin{aligned} |f| &= f(s, V) \\ &= f(s, V) + f(S - \{s\}, V) \qquad [t \notin S \Rightarrow f(S - \{s\}, V) = 0] \\ &= f(S, V) \\ &= f(S, T) + f(S, S) \\ &= f(S, T). \end{aligned}$$

Cuts and Flows (cont'd)

Corollary 12

The value of any flow in a network is bounded from above by the capacity of any cut.

Proof: Let f be a flow and (S, T) a cut. Then

$$|f| = f(S, T) \leq c(S, T).$$

The Max-Flow Min-Cut Theorem

Theorem 13

Let $\mathcal{N} = (\mathcal{G} = (V, E), c, s, t)$ be a flow network.
Then the maximum value of a flow in $\mathcal{N}$ is equal to the minimum capacity of a cut in $\mathcal{N}$.

Proof of the Max-Flow Min-Cut Theorem

Let f be a flow of maximum value and (S, T) a cut of minimum capacity in $\mathcal{N}$. We shall prove that

$$|f| = c(S, T).$$

1. $|f| \leq c(S, T)$ follows from Corollary 12.
   So all we have to prove is that there is a cut (S, T) such that
   $$c(S, T) \leq |f|.$$
2. First remember that $|f|$ has no augmenting path.
   Proof: If $\mathcal{P}$ was an augmenting path, then $f + f_{\mathcal{P}}$ would be a flow of larger value (because by definition of $\mathcal{N}_f$, all edges in $\mathcal{N}_f$ have strictly positive weights).
3. Thus there is no path from s to t in $\mathcal{N}_f$. Let
   $$S = \{v \mid \text{there is a path from } s \text{ to } v \text{ in } \mathcal{N}_f\}$$
   and $T = V \setminus S$. Then (S, T) is a cut.

Proof of the Max-Flow Min-Cut Theorem (cont'd)

4. By definition of S, and because reachability in graphs is a transitive relation, there cannot be any edge from S to T in $\mathcal{N}_f$. Thus for all $u \in S$, $v \in T$ we have $c(u,v) - f(u,v) = 0$.
5. Thus
$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v) = \sum_{u \in S} \sum_{v \in T} f(u,v) = f(S,T) = |f|$$
(by Lemma 11).

Corollaries

Corollary 14

A flow is maximum if, and only if, it has no augmenting path.

Proof: This follows from the proof of the Max-Flow Min-Cut theorem.

Corollary 15

If the Ford-Fulkerson algorithm terminates, then it returns a maximum flow.

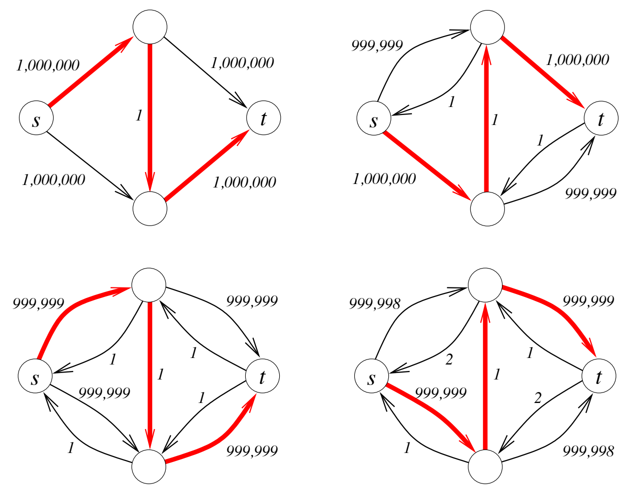
Proof: The flow returned by Ford-Fulkerson has no augmenting path.

Termination

Let f^* be a maximum flow in a network $\mathcal{N}$.

- If all capacities are integers, then Ford-Fulkerson stops after at most
$$|f^*|$$
iterations of the main loop.
- If all capacities are rationals, then Ford-Fulkerson stops after at most
$$q \cdot |f^*|$$
iterations of the main loop, where q is the least common multiple of the denominators of all the capacities.
- For arbitrary real capacities, it may happen that Ford-Fulkerson does not stop.

A Nasty Example



The Edmonds-Karp Heuristic

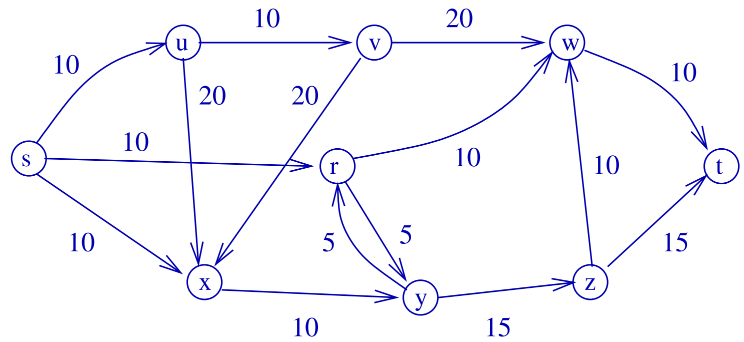
Idea

Always choose a shortest augmenting path.

n number of vertices, m number of edges. Recall that $n \leq m + 1$
A shortest augmenting path can be found by Breadth-First-Search (reading assignment) in time $O(n + m) = O(m)$.

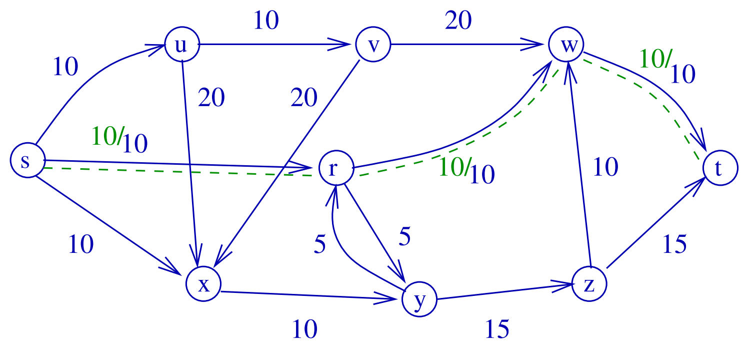
Theorem 16
The Ford-Fulkerson algorithm with the Edmonds-Karp heuristic stops after at most $O(nm)$ iterations of the main loop.
Thus the running time is $O(nm^2)$.

Interesting Example



We will run Ford-Fulkerson (with the Edmonds-Karp heuristic) on this network. This is interesting because we will see the "back-edges" being used to "undo" part of an previous augmenting path.

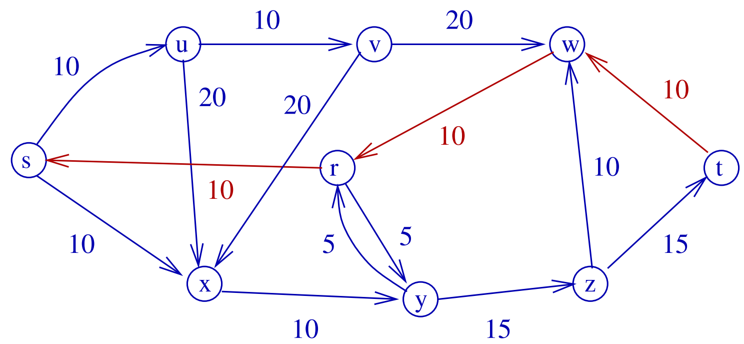
Interesting Example cont.



1st augmenting path: $s \to r \to w \to t$.

Length is 3 (so we satisfy Edmonds-Karp rule to take a shortest possible path). Min capacity is 10, so we push flow of 10 along the path. Starting flow becomes 10.

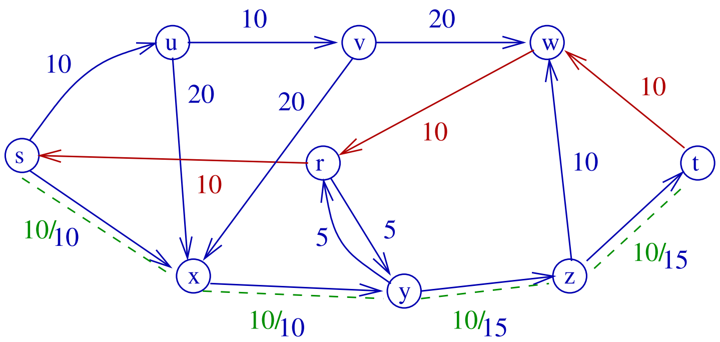
Interesting Example cont.



Residual network after adding first flow of value 10 along $s \to r \to w \to t$.

The newly-created "back-edges" are shown in red.

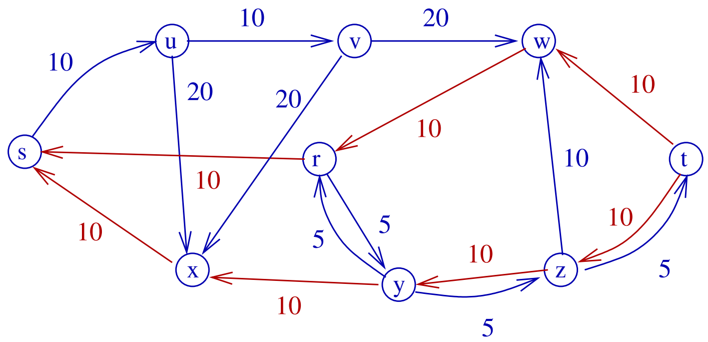
Interesting Example cont.



There is no longer any augmenting path of length ≤ 3, and the only one of length 4 is $s \longrightarrow x \longrightarrow y \longrightarrow z \longrightarrow t$, which has a minimum capacity $\min\{10, 10, 15, 15\}$, ie 10.

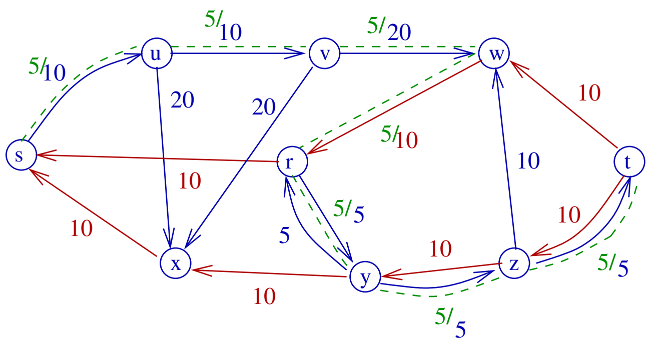
We push this extra flow of value 10 along $s \longrightarrow x \longrightarrow y \longrightarrow z \longrightarrow t$, bringing overall flow to 20.

Interesting Example cont.



Residual network after adding flow from second augmenting path $s \longrightarrow x \longrightarrow y \longrightarrow z \longrightarrow t$, overall flow now 20.

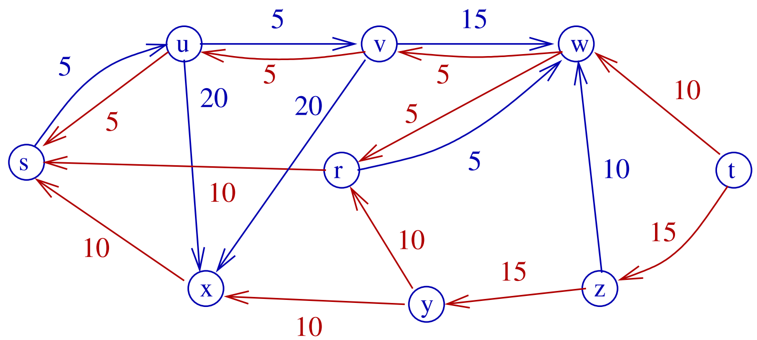
Interesting Example cont.



Now there is only one simple augmenting path - $s \longrightarrow u \longrightarrow v \longrightarrow w \longrightarrow r \longrightarrow y \longrightarrow z \longrightarrow t$, with minimum residual capacity 5.

Notice we use the "back-edge" $w \longrightarrow r$ in our path. This is essentially "re-shipping" 5 units from the first flow-path away from $r \longrightarrow w \longrightarrow t$ and along $r \longrightarrow y \longrightarrow z \longrightarrow t$ instead.

Interesting Example



Residual network after adding 3rd flow, of value 5 $\Rightarrow$ total flow 25.

There is no longer *any* augmenting path in our residual network (set of vertices "reachable" from s is $\{s, u, v, x, w, r\}$).

Reading and Problems

[CLRS] Chapter 26
For breadth-first search: [CLRS], Section 22.2.

Problems

1. Exercise 26.1-5 of [CLRS] (ed 2).

   Not in [CLRS] (ed 3). Question is: consider Figure 26.1(b) and find a pair of subsets $X, Y \subseteq V$ such that $f(X, Y) = -f(V \setminus X, Y)$. After that, find a pair of subsets $X', Y' \subseteq V$ for which $f(X', Y') \neq -f(V \setminus X', Y')$.
2. Exercise 26.2-2 of [CLRS] (2nd ed), Ex 26.2-3 of [CLRS] (3rd ed).
3. Prove Lemma 8.
4. Problem 26-4 of [CLRS].