Algorithms and Data Structures: Counting sort and Radix sort

Special Cases of the Sorting Problem

In this lecture we assume that the sort keys are sequences of bits.

- ► Quite a natural special case. Doesn't cover everything:
 - eg, exact real number arithmetic doesn't take this form.
 - In certain applications, e.g. in Biology, pairwise experiments may only return > or < (non-numeric).
- Sometimes the bits are naturally grouped, e.g. as characters in a string or hexadecimal digits in a number (4 bits), or in general bytes (8 bits).
- Today's sorting algorithms are allowed access these bits or groups of bits, instead of just letting them compare keys ... This was NOT possible in comparison-based setting.

ADS: lect 9 - slide 1 -

Easy results ... Surprising results

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Surprising case: (I think)

For any constant k, the problem of sorting n integers in the range $\{1, ..., n^k\}$ can be done in $\Theta(n)$ time.

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Counting Sort

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Idea

- 1. Count for every key j, $1 \le j \le m$ how often it occurs in the input array. Store results in an array C.
- 2. The counting information stored in C can be used to determine the position of each element in the sorted array. Suppose we modify the values of the C[j] so that now

C[j] = the number of keys *less than or equal* to j.

Then we know that the elements with key "j" must be stored at the indices $C[j-1] + 1, \ldots, C[j]$ of the final sorted array.

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Implementation of Counting Sort

Algorithm COUNTING SORT(A, m)

1. $n \leftarrow A$.length 2. Initialise array $C[1 \dots m]$ for $i \leftarrow 1$ to n do 3. $j \leftarrow A[i]$.key 4. $C[j] \leftarrow C[j] + 1$ 5. 6. for $i \leftarrow 2$ to m do $C[j] \leftarrow C[j] + C[j-1] \quad \rhd \ C[j] \text{ stores } \sharp \text{ of keys} \leq j$ 7. Initialise array $B[1 \dots n]$ 8. for $i \leftarrow n$ downto 1 do 9. 10. $j \leftarrow A[i]$.key > A[i] highest w. key j $B[C[j]] \leftarrow A[i]$ \triangleright Insert A[i] into highest free index for j11. $C[j] \leftarrow C[j] - 1$ 12. **13**. for $i \leftarrow 1$ to n do 14. $A[i] \leftarrow B[i]$

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Then we know that the elements with key "j" must be stored at the indices $C[j-1] + 1, \ldots, C[j]$ of the final sorted array.

3. We use a "trick" to move the elements to the right position of an auxiliary array. Then we copy the sorted auxiliary array back to the original one.

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Analysis of Counting Sort

- The loops in lines 3–5, 9–12, and 13–14 all require time $\Theta(n)$.
- The loop in lines 6–7 requires time $\Theta(m)$.
- Thus the overall running time is

O(n+m).

• This is *linear* in the number of elements if m = O(n).

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Radix Sort

Basic Assumption

Keys are sequences of digits in a fixed range $0, \ldots, R-1$, all of equal length d.

Examples of such keys

- 4 digit hexadecimal numbers (corresponding to 16 bit integers)
 R = 16, d = 4
- 5 digit decimal numbers (for example, US post codes)
 R = 10, d = 5
- Fixed length ASCII character sequences
 R = 128
- ► Fixed length byte sequences *R* = 256

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Note: COUNTING-SORT is STABLE.

(After sorting, 2 items with the same key have their *initial relative* order).

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Stable Sorting Algorithms

Definition 1

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Examples

- COUNTING-SORT, MERGE-SORT, and INSERTION SORT are all stable.
- ► QUICKSORT is not stable.
- If keys and elements are exactly the same thing (in our setting, an element is a structure containing the key as a sub-element) then we have a much easier (non-stable) version of COUNTING-SORT. (How? ... CLASS?).

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Radix Sort (cont'd)

Algorithm RADIX-SORT(A, d)

1. for $i \leftarrow 0$ to d do

2. use stable sort to sort array A using digit *i* as key

Most commonly, COUNTING SORT is used in line 2 - this means that once a set of digits is already in sorted order, then (by stability) performing COUNTING SORT on the *next-most significant* digits preserves that order, within the "blocks" constructed by the new iteration.

Then each execution of line 2 requires time $\Theta(n+R)$. Thus the overall time required by RADIX-SORT is

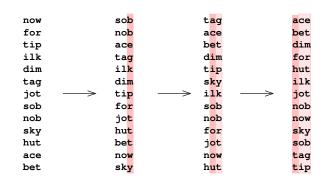
$$\Theta(d(n+R))$$

Radix Sort (cont'd)

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Sort the keys digit by digit, *starting with the least significant digit*.

Example



Each of the three sorts is carried out with respect to the digits in that column. "Stability" (and having previously sorted digits/suffixes to the right), means this achieves a sorting of the suffixes starting at the current column. ADS: lect 9 - slide 9 - blick lect 9 - blick lect 9 - slide 9 - blick lect 9 - blick lect 9 - slide 9 - blick lect 9 - blick 1 - blick

Sorting Integers with Radix-Sort

Theorem 2

An array of length n whose keys are b-bit numbers can be sorted in time

$$\Theta(n\lceil b/\lg n\rceil)$$

using a suitable version of RADIX-SORT.

Proof: Let the digits be blocks of $\lceil \lg n \rceil$ bits. Then $R = 2^{\lceil \lg n \rceil} = \Theta(n)$ and $d = \lceil b / \lceil \lg n \rceil \rceil$. Using the implementation of RADIX-SORT based on COUNTING SORT the integers can be sorted in time

$$\Theta(d(n+R)) = \Theta(n\lceil b/\lg n\rceil).$$

Note: If all numbers are at most n^k , then $b = k \lg n \ldots \Rightarrow$ Radix Sort is $\Theta(n)$ (assuming k is some constant, eg 3, 10).

Reading Assignment

[CLRS] Sections 8.2, 8.3

Problems

- 1. Think about the qn. on slide 7 how do we get a very easy (non-stable) version of COUNTING-SORT if there are no items attached to the keys?
- 2. Can you come up with another way of achieving counting sort's O(m + n)-time bound and stability (you will need a different data structure from an array).
- 3. Exercise 8.3-4 of [CLRS].

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