Algorithms and Data Structures
Strassen's Algorithm

Tutorials

- Start in week (week 3)
- Tutorial allocations are linked from the course webpage
  `http://www.inf.ed.ac.uk/teaching/courses/ads/`

The Master Theorem for solving recurrences

Theorem

Let $n_0 \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $a, b \in \mathbb{R}$ with $a > 0$ and $b > 1$, and let $T : \mathbb{N} \to \mathbb{R}$ satisfy the following recurrence:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n < n_0, \\ a \cdot T(n/b) + \Theta(n^k) & \text{if } n \geq n_0. \end{cases}$$

Let $c = \log_b(a)$; we call c the critical exponent. *Then*

$$T(n) = \begin{cases} \Theta(n^c) & \text{if } k < c & \text{(I)}, \\ \Theta(n^c \cdot \lg(n)) & \text{if } k = c & \text{(II)}, \\ \Theta(n^k) & \text{if } k > c & \text{(III)}. \end{cases}$$

Theorem also holds if we replace $a \cdot T(n/b)$ above by $a_1 \cdot T(\lfloor n/b \rfloor) + a_2 \cdot T(\lceil n/b \rceil)$ for any $a_1, a_2 \geq 0$ with $a_1 + a_2 = a$.

The Master Theorem (cont'd)

- We don't have time to prove the Master Theorem in class. You can find the proof in Section 4.6 of [CLRS]. *Section 4.4 of [CLRS], 2nd ed*.
  Their version of the M.T. is a bit more general than ours.
- Consider the following examples:

$$
\begin{aligned}
T(n) &= 4T(n/2) + n, \\
T(n) &= 4T(\lfloor n/2 \rfloor) + n^2, \\
T(n) &= 4T(n/2) + n^3.
\end{aligned}
$$

  Could alternatively unfold-and-sum to prove the first and third of these (and to get an estimate for the second).

CLASS EXERCISE

Matrix Multiplication

Recall

The product of two $(n \times n)$-matrices

$$A = (a_{ij})_{1 \leq i,j \leq n} \quad \text{and} \quad B = (b_{ij})_{1 \leq i,j \leq n}$$

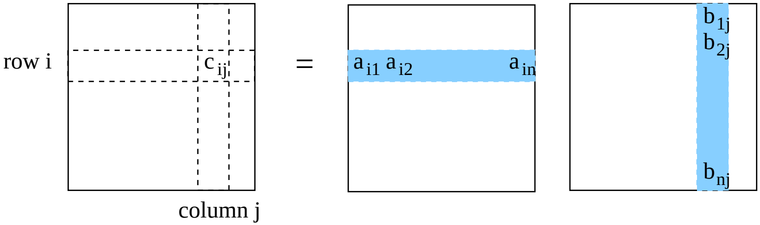
is the $(n \times n)$-matrix $C = AB$ where $C = (c_{ij})_{1 \leq i,j \leq n}$ with entries

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

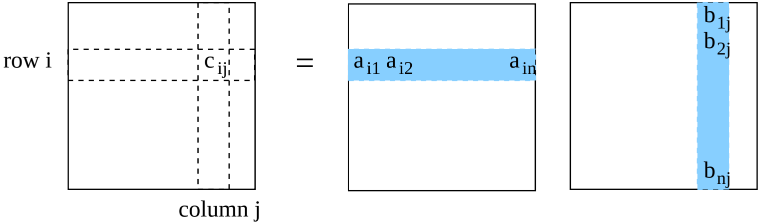
The Matrix Multiplication Problem

Input: $(n \times n)$-matrices A and B
Output: the $(n \times n)$-matrix AB

Matrix Multiplication



Matrix Multiplication



- n multiplications and n additions for each c_{ij}.
- there are n^2 different c_{ij} entries.

A straightforward algorithm

Algorithm $\textsc{MatMult}(A, B)$

```
n ← number of rows of A
for i ← 1 to n do
    for j ← 1 to n do
        c_ij ← 0
        for k ← 1 to n do
            c_ij ← c_ij + a_ik · b_kj
return C = (c_ij)_{1≤i,j≤n}
```

Requires

$$\Theta(n^3)$$

arithmetic operations (additions and multiplications).

A näive divide-and-conquer algorithm

Observe

If

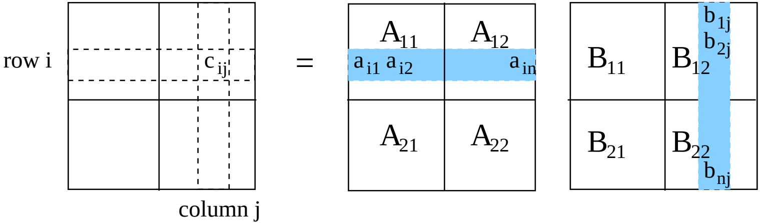
$$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \quad \text{and} \quad B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$$

for $(n/2 \times n/2)$-submatrices A_{ij} and B_{ij} then

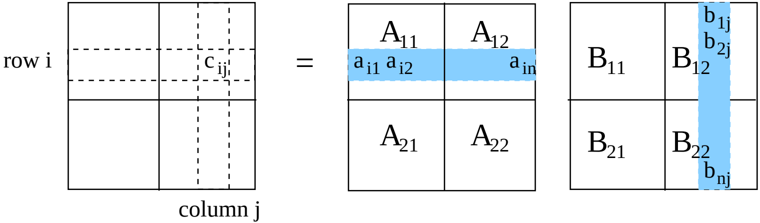
$$AB = \left(\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right)$$

note: We are assuming n is a power of 2.

A näive divide-and-conquer algorithm



A näive divide-and-conquer algorithm



Suppose $i \leq n/2$ and $j > n/2$. Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \underbrace{\sum_{k=1}^{n/2} a_{ik} b_{kj}}_{\in A_{11} B_{12}} + \underbrace{\sum_{k=n/2+1}^{n} a_{ik} b_{kj}}_{\in A_{12} B_{22}}$$

A näive divide-and-conquer algorithm (cont'd)

Assume n is a power of 2.

Algorithm $\textsc{D\&C-MatMult}(A, B)$

1. $n \leftarrow$ number of rows of A
2. **if** $n = 1$ **then return** $(a_{11}b_{11})$
3. **else**
4. Let A_{ij}, B_{ij} (for $i, j = 1, 2$) be $(n/2 \times n/2)$-submatrices s.th.
   $A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)$ and $B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$
5. Recursively compute $A_{11}B_{11}$, $A_{12}B_{21}$, $A_{11}B_{12}$, $A_{12}B_{22}$,
   $A_{21}B_{11}$, $A_{22}B_{21}$, $A_{21}B_{12}$, $A_{22}B_{22}$
6. Compute $C_{11} = A_{11}B_{11} + A_{12}B_{21}$, $C_{12} = A_{11}B_{12} + A_{12}B_{22}$,
   $C_{21} = A_{21}B_{11} + A_{22}B_{21}$, $C_{22} = A_{21}B_{12} + A_{22}B_{22}$
7. **return** $\left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right)$

Analysis of D&C-MatMult

$T(n)$ is the number of operations done by D&C-MatMult.

- Lines $1, 2, 3, 4, 7$ require $\Theta(1)$ arithmetic operations
- Line 5 requires $8T(n/2)$ arithmetic operations
- Line 6 requires $4(n/2)^2 = \Theta(n^2)$ arithmetic operations.
  Remember! Size of matrices is $\Theta(n^2)$, NOT $\Theta(n)$

We get the recurrence

$$T(n) = 8T(n/2) + \Theta(n^2).$$

Since $\log_2(8) = 3$, the Master Theorem yields

$$T(n) = \Theta(n^3).$$

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(No improvement over MatMult ... why? CLASS? ...)

Strassen's algorithm (1969)

Assume n is a power of 2.

Let

$$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \quad \text{and} \quad B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right).$$

We want to compute

$$\begin{aligned} AB &= \left(\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right) \\ &= \left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right). \end{aligned}$$

Strassen's algorithm uses a *trick* in applying Divide-and-Conquer.

Strassen's algorithm (cont'd)

Let

$$
\begin{aligned}
P_1 &= (A_{11} + A_{22})(B_{11} + B_{22}) \\
P_2 &= (A_{21} + A_{22})B_{11} \\
P_3 &= A_{11}(B_{12} - B_{22}) \\
P_4 &= A_{22}(-B_{11} + B_{21}) \\
P_5 &= (A_{11} + A_{12})B_{22} \\
P_6 &= (-A_{11} + A_{21})(B_{11} + B_{12}) \\
P_7 &= (A_{12} - A_{22})(B_{21} + B_{22})
\end{aligned}
\qquad (*)
$$

Strassen's algorithm (cont'd)

Let

$$\begin{aligned}
P_1 &= (A_{11} + A_{22})(B_{11} + B_{22}) \\
P_2 &= (A_{21} + A_{22})B_{11} \\
P_3 &= A_{11}(B_{12} - B_{22}) \\
P_4 &= A_{22}(-B_{11} + B_{21}) \\
P_5 &= (A_{11} + A_{12})B_{22} \\
P_6 &= (-A_{11} + A_{21})(B_{11} + B_{12}) \\
P_7 &= (A_{12} - A_{22})(B_{21} + B_{22})
\end{aligned} \tag{$*$}$$

Then

$$\begin{aligned}
C_{11} &= P_1 + P_4 - P_5 + P_7 & \qquad C_{12} &= P_3 + P_5 \\
C_{21} &= P_2 + P_4 & \qquad C_{22} &= P_1 + P_3 - P_2 + P_6
\end{aligned} \tag{$**$}$$

Checking Strassen's algorithm - $C11$

We will check the equation for C_{11} is correct.
Strassen's algorithm computes $C_{11} = P1 + P4 - P5 + P7$. We have

$$
\begin{aligned}
P1 &= (A11 + A22)(B11 + B22) \\
&= A11B11 + A11B22 + A22B11 + A22B22. \\
P4 &= A22(-B11 + B21) \ = \ A22B21 - A22B11. \\
P5 &= (A11 + A12)B22 \ = \ A11B22 + A12B22. \\
P7 &= (A12 - A22)(B21 + B22) \\
&= A12B21 + A12B22 - A22B21 - A22B22.
\end{aligned}
$$

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&= A12B21 + A12B22 - A22B21 - A22B22.
\end{aligned}$$

Then $P1 + P4 = A11B11 + A11B22 + A22B22 + A22B21$.

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P5 &= (A11 + A12)B22 \;=\; A11B22 + A12B22. \\
P7 &= (A12 - A22)(B21 + B22) \\
&= A12B21 + A12B22 - A22B21 - A22B22.
\end{aligned}
$$

Then $P1 + P4 = A11B11 + A11B22 + A22B22 + A22B21$.
Then $P1 + P4 - P5 = A11B11 + A22B22 + A22B21 - A12B22$.

Checking Strassen's algorithm - $C11$

We will check the equation for C_{11} is correct.
Strassen's algorithm computes $C_{11} = P1 + P4 - P5 + P7$. We have

$$\begin{aligned}
P1 &= (A11 + A22)(B11 + B22) \\
&= A11B11 + A11B22 + A22B11 + A22B22. \\
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P5 &= (A11 + A12)B22 \;=\; A11B22 + A12B22. \\
P7 &= (A12 - A22)(B21 + B22) \\
&= A12B21 + A12B22 - A22B21 - A22B22.
\end{aligned}$$

Then $P1 + P4 = A11B11 + A11B22 + A22B22 + A22B21$.
Then $P1 + P4 - P5 = A11B11 + A22B22 + A22B21 - A12B22$.
Then $P1 + P4 - P5 + P7 = A11B11 + A12B21$, which is $C11$.

Checking Strassen's algorithm - $C11$

We will check the equation for C_{11} is correct.
Strassen's algorithm computes $C_{11} = P1 + P4 - P5 + P7$. We have

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P5 &= (A11 + A12)B22 \; = \; A11B22 + A12B22. \\
P7 &= (A12 - A22)(B21 + B22) \\
&= A12B21 + A12B22 - A22B21 - A22B22.
\end{aligned}
$$

Then $P1 + P4 = A11B11 + A11B22 + A22B22 + A22B21$.
Then $P1 + P4 - P5 = A11B11 + A22B22 + A22B21 - A12B22$.
Then $P1 + P4 - P5 + P7 = A11B11 + A12B21$, which is $C11$.

homework: check other 3 equations.

Strassen's algorithm (cont'd)

Crucial Observation

Only **7** multiplications of $(n/2 \times n/2)$-matrices are needed to compute AB.

Algorithm $\textsc{Strassen}(A, B)$

1. $n \leftarrow$ number of rows of A
2. **if** $n = 1$ **then return** $(a_{11} b_{11})$
3. **else**
4. Determine A_{ij} and B_{ij} for $i, j = 1, 2$ (as before)
5. Compute $P_1, \ldots, P_7$ as in $(*)$
6. Compute $C_{11}, C_{12}, C_{21}, C_{22}$ as in $(**)$
7. **return** $\left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right)$

Analysis of Strassen's algorithm

Let $T(n)$ be the number of arithmetic operations performed by STRASSEN.

- Lines $1-4$ and 7 require $\Theta(1)$ arithmetic operations
- Line 5 requires $7T(n/2) + \Theta(n^2)$ arithmetic operations
- Line 6 requires $\Theta(n^2)$ arithmetic operations. remember.

We get the recurrence

$$T(n) = 7T(n/2) + \Theta(n^2).$$

Since $\log_2(7) \approx 2.807 > 2$, the Master Theorem yields

$$T(n) = \Theta(n^{\log_2(7)}).$$

Breakthroughs on matrix multiplication

- Coppersmith & Winograd (1987) came up with an improved algorithm with running time of

$$\Theta(n^{2.376}).$$

- *... many years of silence ...*
- Then in his 2010 PhD thesis, **Andrew Stothers** from the School of Maths, at the **University of Edinburgh** got an algorithm with $\Theta(n^c)$ for $c < 2.3737$ ...
  - $\Rightarrow$ Coppersmith/Winograd not optimal.
  - But Stothers didn't publish.
- In December 2011, Virginia Vassilevska Williams of Stanford, came up with a $\Theta(n^c)$ algoithm, for $c < 2.3727$ (partly, but not only, making use of some of Stothers' ideas)

Remarks on Matrix Multiplication

- In practice, the "school" MatMult algorithm tends to outperform Strassen's algorithm, unless the matrices are huge.
- The best known lower bound for matrix multiplication is

$$\Omega(n^2).$$

This is a *trivial* lower bound (need to look at all entries of each matrix). Amazingly, $\Omega(n^2)$ is believed to be "the truth"!

Open problem: Can we find a $O(n^{2+o(1)})$-algorithm for Matrix Multiplication of $n \times n$ matrices?

Reading Assignment

[CLRS] (3rd ed) Section 4.5 "The Master method for solving recurrences"
(*Section 4.3 "Using the Master method" of [CLRS], 2nd ed*)
[CLRS] (3rd ed) Section 4.2 (*Section 28.2 of [CLRS], 2nd ed*)

Problems

1. Exercise 4.5-2 of [CLRS] (3rd ed) *Exercise 4.3-2 of [CLRS], 2nd ed.*
2. Exercise 4.2-1 of [CLRS], 3rd ed. *Exercise 28.2-1 [CLRS], 2nd ed.*
3. Week 3 tutorial sheet :-)