Algorithms and Data Structures:
Minimum Spanning Trees (Kruskal)
Minimum Spanning Tree Problem

Given: *Undirected connected weighted graph* \((G, W)\)
Output: An MST of \(G\)

- We have already seen the PRIM algorithm, which runs in \(O((m + n) \log(n))\) time (standard Heap implementation) for graphs with \(n\) vertices and \(m\) edges.
- In this lecture we will see KRUSKAL’s algorithm, a different approach to constructing a MST.
Kruskal’s Algorithm

A forest is a graph whose connected components are trees.

Idea
Starting from a spanning forest with no edges, repeatedly add edges of minimum weight (never creating a cycle) until the forest becomes a tree.

Algorithm \textsc{Kruskal}(G, W)

1. \( F \leftarrow \emptyset \)
2. \text{for all } e \in E \text{ in the order of increasing weight do}
3. \quad \text{if the endpoints of } e \text{ are in different connected components of } (V, F) \text{ then}
   \quad F \leftarrow F \cup \{e\}
4. \text{return tree with edge set } F
Example
Correctness of Kruskal’s algorithm

1. Throughout the execution of KRUSKAL, \((V, F)\) remains a spanning forest.
   \(Proof:\) \((V, F)\) is a spanning subgraph because the vertex set is \(V\). It always remains a forest because edges with endpoints in different connected components never induce a cycle.

2. Eventually, \((V, F)\) will be connected and thus a spanning tree.
   \(Proof:\) Suppose that after the complete execution of the loop, \((V, F)\) has a connected component \((V_1, F_1)\) with \(V_1 \neq V\). Since \(G\) is connected, there is an edge \(e \in E\) with exactly one endpoint in \(V_1\). This edge would have been added to \(F\) when being processed in the loop, so this can never happen.

3. Throughout the execution of KRUSKAL, \((V, F)\) is contained in some MST of \(G\).
   \(Proof:\) Similar to the proof of the corresponding statement for Prim’s algorithm. Will prove in week 9 Tutorial.
Data Structures for Disjoint Sets

- A disjoint set data structure maintains a collection \( S = \{S_1, \ldots, S_k\} \) of disjoint sets.

- The sets are *dynamic*, i.e., they may change over time.

- Each set \( S_i \) is identified by some *representative*, which is some member of that set.

**Operations:**

- **Make-Set** \((x)\): Creates new set whose only member is \( x \). The representative is \( x \).

- **Union** \((x, y)\): Unites set \( S_x \) containing \( x \) and set \( S_y \) containing \( y \) into a new set \( S \) and removes \( S_x \) and \( S_y \) from the collection.

- **Find-Set** \((x)\): Returns representative of the set holding \( x \).
Implementation of Kruskal’s Algorithm

Algorithm Kruskal(G, W)
1. $F \leftarrow 0$
2. for all vertices $v$ of $G$ do
3.     Make-Set($v$)
4. sort edges of $G$ into non-decreasing order by weight
5. for all edges $(u, v)$ of $G$ in non-decreasing order by weight do
6.     if Find-Set($u$) $\neq$ Find-Set($v$) then
7.         $F \leftarrow F \cup \{(u, v)\}$
8.     Union($u$, $v$)
9. return $F$
Analysis of Kruskal

Let $n$ be the number of vertices and $m$ the number of edges of the input graph.

- Line 1: $\Theta(1)$
- Loop in Lines 2–3: $\Theta(n \cdot T_{\text{MAKE-SET}}(n))$
- Line 4: $\Theta(m \log m)$
- Loop in Lines 5–8: $\Theta\left(2m \cdot T_{\text{FIND-SET}}(n) + (n - 1) \cdot T_{\text{UNION}}(n)\right)$
- Line 9: $\Theta(1)$

Overall:

$$\Theta\left(nT_{\text{MAKE-SET}}(n) + (n - 1)T_{\text{UNION}}(n) + m(\log m + 2T_{\text{FIND-SET}}(n))\right)$$
Analysis of **Kruskal** (overview)

\[ T(n, m) = \Theta \left( n T_{\text{MAKE-SET}}(n) + (n-1) T_{\text{UNION}}(n) + m \left( \lg m + 2 T_{\text{FIND-SET}}(n) \right) \right) \]

We will see that with standard efficient implementations of disjoint sets this amounts to

\[ T(n, m) = \Theta(m \lg(m)). \]

- **NOT** better than the standard Heap implementation of **Prim** for typical implementations of disjoint sets.
- Always have to sort the weights when using **Kruskal**:
  - \( \Theta(m \lg(m)) \) if the weights are arbitrarily large.
Linked List Implementation of Disjoint Sets

Each element represented by a pointer to a cell:

Use a linked list for each set.
Representative of the set is at the head of the list.
Each cell has a pointer direct to the representative (head of the list).
Example

Linked list representation of

\{a, f\}, \quad \{b\}, \quad \{g, c, e\}, \quad \{d\}:

The "representatives" are a, b, g and d respectively.

last( ) pointers are in red.
Analysis of Linked List Implementation

**MAKE-SET**: constant ($\Theta(1)$) time.

**FIND-SET**: constant ($\Theta(1)$) time.

**UNION**: Naive implementation of

\[ \text{UNION}(x, y) \]

appends $x$’s list onto end of $y$’s list.

**Assumption**: Representative $y$ of each set has attribute $\text{last}[y]$: a pointer to last cell of $y$’s list.

**Snag**: have to update “representative pointer” in each cell of $x$’s list to point to the representative (head) of $y$’s list. Cost is:

\[ \Theta(\text{length of } x\text{’s list}). \]
Notation for Analysis

Express running time in terms of:

\( \hat{n} \): the number of `MAKE-SET` operations,
\( \hat{m} \): the number of `MAKE-SET`, `UNION` and `FIND-SET` operations overall.

Note

1. After \( \hat{n} - 1 \) `UNION` operations only one set remains.
2. \( \hat{m} \geq \hat{n} \).
Weighted-Union Heuristic

Idea

Maintain a “length” field for each list. To execute

\[
\text{\textsc{Union}}(x, y)
\]

append shorter list to longer one (breaking ties arbitrarily).

Theorem 1

Using the linked-list representation of disjoint sets and the weighted-union heuristic, a sequence of \( \hat{m} \) \textsc{Make-Set}, \textsc{Union} \& \textsc{Find-Set} operations, \( \hat{n} \) of which are \textsc{Make-Set} operations, takes

\[
O(\hat{m} + \hat{n} \lg \hat{n})
\]

time.

“Proof”: Each element appears at most \( \lg \hat{n} \) times in the short list of a \textsc{Union}. 

ADS: lecture 16 – slide 14 –
Example \((\text{UNION}(g, b)))\)

- b's "representative" pointer changes to point at g-cell
- e's "next" pointer changes to point at b-cell
- g's "last" pointer changes to point at b-cell

b's list is shorter than g's, so it joins the end of g's list

result of performing \(\text{UNION}(g, b)\)
Kruskal with Linked lists (weighted union)

The run-time for \textsc{Kruskal} (for $\mathcal{G} = (V, E)$ with $|V| = n, |E| = m$) is

$$T(n, m) = \Theta\left(n T_{\text{MAKE-SET}}(n) + (n-1) T_{\text{UNION}}(n) + m\left(\lg m + 2 T_{\text{FIND-SET}}(n)\right)\right)$$

In terms of the collection of “Disjoint-sets” operations, we have $\hat{m} = 2n + 2m - 1$ operations, $\hat{n} = n$ which are \textsc{Union}. So

$$T(n, m) = \Theta(m \lg(m) + (2n + 2m - 1) + n \lg(n)) = \Theta(m \lg(m))$$