One-Unambiguous Regular Languages

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The ISO standard for the Standard Generalized Markup Language (SGML) provides a syntactic meta-language for the definition of textual markup systems. In the standard, the right-hand sides of productions are based on regular expressions, although only regular expressions that denote words unambiguously, in the sense of the ISO standard, are allowed. In general, a word that is denoted by a regular expression is witnessed by a sequence of occurrences of symbols in the regular expression that match the word. In an unambiguous regular expression as defined by Book et al. (1971, IEEE Trans. Comput. C-20(2), 149–153), each word has at most one witness. But the SGML standard also requires that a witness be computed incrementally from the word with a one-symbol lookahead; we call such regular expressions 1-unambiguous. A regular language is a 1-unambiguous language if it is denoted by some 1-unambiguous regular expression. We give a Kleene theorem for 1-unambiguous languages and characterize 1-unambiguous regular languages in terms of structural properties of the minimal deterministic automata that recognize them. As a result we are able to prove the decidability of whether a given regular expression denotes a 1-unambiguous language; if it does, then we can construct an equivalent 1-unambiguous regular expression in worst-case optimal time. © 1998

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1. INTRODUCTION

Document processing systems such as editors, formatters, and retrieval systems deal with many different types of documents, such as books, articles, memoranda, dictionaries, and letters. The Standard Generalized Markup Language (SGML) establishes a common platform for the syntactic specification of document types and conforming documents [ISO86, Gol90]. SGML is an ISO standard that has been endorsed by a number of publishing houses throughout North America and Europe, by the European Community, and by the U.S. Department of Defense.

Document types in SGML are defined, essentially, by bracketed, extended context-free grammars [GH67, Woo87]. The right-hand sides of productions, called model groups, are essentially regular expressions with two major differences. First, model groups allow three new operators ?, & , and +. Second, the model groups must be unambiguous in the sense of Clause 11.2.4.3 of the standard. The intent of the standard is to make it easier for a human to write regular expressions that can be interpreted unambiguously. The notion of unambiguity for model groups used in the ISO standard differs from, but is related to, unambiguity as defined by Book et al. [BEGO71] for regular expressions. Eilenberg [Eil74] defined ambiguity for finite-state automata, rather than for regular expressions; we discuss the relationship of his notion to the one of Book et al. at the end of Section 2. Book et al. required that each words be witnessed by at most one sequence of positions of symbols in the regular expression that matches the word. For example, consider the regular expression \((a+b)^* aa^*\). If we mark different positions of the same symbol with subscripts, we get \((a_1+b_1)^* a_2a_3^*\); now, there are three witnesses for the word \(aaa\), namely \(a_1a_1a_2, a_1a_2a_3, \) and \(a_2a_3a_3\). Thus, \((a+b)^* aa^*\) is ambiguous; however, there is an unambiguous regular expression that denotes the same language, namely \((a+b)^* a\).

Unambiguity as defined in SGML is a one-symbol-lookahead version of unambiguity as defined by Book et al. In Clause 11.2.4.3 of the SGML standard a model group (regular expression) is defined to be unambiguous if \(^1\) “an element [a symbol]... that occurs in the document instance [word] must be able to satisfy only one primitive content token [position of the symbol in the regular expression] without looking ahead in the document instance.” In other words, the only valid regular expressions are those that permit us to determine uniquely which position of a symbol in a regular expression should match a symbol in an input word without looking beyond that symbol in the input word. We call such regular expressions 1-unambiguous.

Consider the regular expression \((a+b)^* a\) marked as \((a_1+b_1)^* a_2\). In the word \(baa\), after we match symbol \(b\) with position \(b_1\), we cannot decide whether we should match the subsequent \(a\) in the word with position \(a_1\) or with position \(a_2\) without looking ahead beyond the current symbol \(a\) in the word. Therefore, although \((a+b)^* a\) is unambiguous in the sense of Book et al., it is not 1-unambiguous; however, \(b^*a(b^*a)^*\) is a 1-unambiguous regular expression that denotes the same language as \((a+b)^* a\).

\(^1\) The phrases in brackets are our interpretations
For unambiguous regular expressions in the sense of Book et al. [BEGO71], the following results are known: Book et al. gave a construction that, for each regular expression $E$, gives a nondeterministic finite-state automaton (NFA) $G_E$ that recognizes the language of $E$. They show that $E$ is unambiguous if and only if $G_E$ is unambiguous. Berry and Sethi [BS86] showed that this NFA is the canonical representation of the corresponding regular expression, because it has a natural connection with the derivatives [Brz64] of the regular expression.

Regular expressions are built with the usual operators $+, \cdot$, and $\ast$. SGML, however, deals with model groups that may also contain the operators $\sqcup$, $\sqcap$, and $\ast$. $(E? \sqcup F \sqcap G)$ denotes $L(E + FG + GF)$, and $E^\ast$ denotes $L(EE^\ast)$. Whereas the transformations of $E^\ast$ into $EE^\ast$ and of $F \sqcap G$ into $FG + GF$ preserve languages, they do not preserve 1-ambiguity. For example, $(a^\ast + b)^\ast$ is 1-unambiguous, but $(a^\ast + b)(a^\ast + b)^\ast$ is not. Similarly, $a? \sqcup b$ is 1-unambiguous, but $a?b + ba?$ is not. In fact, there are languages that can be denoted by a 1-unambiguous model group but not by any 1-unambiguous regular expression [BK93a]. Furthermore, 1-unambiguous model groups are exponentially more succinct than are 1-unambiguous regular expressions. For example, the smallest 1-unambiguous regular expression equivalent to the 1-unambiguous model group $a_1^\ast \sqcup \cdots \sqcup a_n^\ast$ has size exponential in $n$.

We establish the basic results for 1-unambiguous regular expressions and languages which are the basis for the results by Ahonen [AH97] for transforming an ambiguous model group into an unambiguous one by generalizing the language of the model group, the decidability results by Brüggemann–Klein [BK93a] for model groups, and the results for SGML exceptions by Kilpeläinen and Wood [KILW97]. In Section 2, after giving the basic definitions, we show that a regular expression $E$ is 1-unambiguous if and only if $G_E$ is a deterministic finite-state automaton (DFA). Thus, one can decide, in time linear in the size of a regular expression $E$, whether $E$ is 1-unambiguous, and if it is, then one can also construct the deterministic automaton $G_E$ in linear time [BK92a, BK93c].

We establish, in Section 3, that the family of 1-unambiguous languages is closed under derivatives and, in Section 4, we establish a Kleene characterization of the family of 1-unambiguous languages. An analogous Kleene characterization for model groups still eludes us.

In Section 5, we present the main result of the paper—a characterization of the 1-unambiguous languages in terms of their minimal deterministic finite-state automata. As one application of this result we prove that the family of 1-unambiguous languages forms a proper subfamily of the regular languages, in contrast to the result of Book et al. that each regular language is denoted by some unambiguous regular expression. The characterization yields a decision algorithm for 1-unambiguous languages. Moreover, if a regular language is 1-unambiguous, then we can construct an equivalent 1-unambiguous regular expression. The decision algorithm runs in time quadratic in the size of the minimal deterministic finite-state automaton for the given language. The 1-unambiguous regular expression that we construct from its minimal deterministic finite-state automaton $M$ can have size exponential in the size of $M$, which is worst-case optimal.
2. UNAMBIGUOUS REGULAR EXPRESSIONS

Let \( \Sigma \) be an alphabet of symbols. Regular expressions over \( \Sigma \) are built from \( \cdot \), \( \emptyset \), and symbols in \( \Sigma \) using the binary operators \( + \) and \( \cdot \) and the unary operator \( ^* \). The language specified by a regular expression \( E \) is denoted by \( L(E) \). The symbols that occur in a regular expression \( E \) are denoted by \( \text{sym}(E) \).

To indicate different positions of the same symbol in a regular expression, we mark symbols with subscripts. For example, \( (a_1 + b_1)^* a_2 (a_3 b_2)^* \) and \( (a_4 + b_2)^* a_1 (a_3 b_1)^* \) are both markings of the regular expression \( (a + b)^* (ab)^* \). For each regular expression \( E \) over \( \Sigma \), a marking of \( E \) is denoted by \( E^* \). If \( H \) is a subexpression of \( E \), we assume that markings \( H^* \) and \( E^* \) are chosen in such a way that \( H^* \) is a subexpression of \( E^* \). A marked regular expression \( E \) is a regular expression over \( \Pi \), the alphabet of subscripted symbols, where each subscripted symbol occurs at most once in \( E \).

The reverse of marking is the dropping of subscripts, indicated by \( \hat{\cdot} \) and defined as follows: If \( E \) is a regular expression over \( \Pi \), then \( E^* \) is the regular expression over \( \Sigma \) that is obtained from \( E \) by dropping all subscripts in \( E \). Thus, a marked regular expression \( H \) is a marking of regular expression \( E \) if and only if \( H^* = E^* \). Unmarking can also be extended to words and languages: For a word \( w \) over \( \Pi \), let \( w^* \) denote the word over \( \Sigma \) that is constructed from \( w \) by dropping all subscripts. For a language \( L \) over \( \Pi \), let \( L^* \) denote \( \{ w^* \mid w \in L \} \). Then, for each regular expression \( E \) over \( \Pi \), \( L(E^*) = L(E)^* \).

Uppercase letters from \( E \) through \( J \) denote regular expressions over \( \Sigma \) or over \( \Pi \); the letters \( a, b, \) and \( c \) denote symbols in \( \Sigma \); the letters \( x, y, \) and \( z \) denote subscripted symbols in \( \Pi \); and the letters \( u, v, \) and \( w \) denote words over \( \Sigma \) or over \( \Pi \).

We now give a concise definition of the SGML notion of unambiguity.

**Definition 2.1.** A regular expression \( E \) is 1-unambiguous if and only if, for all words \( u, v, w \) over \( \Pi \) and all symbols \( x, y \) in \( \Pi \), the conditions \( uxx, uyw \in L(E^*) \) and \( x \neq y \) imply \( x^* \neq y^* \). A regular language is 1-unambiguous if it is denoted by some 1-unambiguous regular expression.

In other words, for each word \( w \) denoted by a 1-unambiguous regular expression \( E \), there is exactly one witness; that is, there is one marked word \( v \) in \( L(E^*) \) such that \( v^* = w \). Furthermore, \( v \) can be constructed incrementally by examining the next symbol of \( w \) that matches the next position of \( v \). It is not hard to see that this definition is independent of the marking \( E^* \) chosen for \( E \). We derive an alternative definition in terms of the pairs of positions that follow each other in a word of \( L(E^*) \).

**Definition 2.2.** For each language \( L \), we define the following four sets:

- \( \text{first}(L) = \{ b \mid \text{there is a word } w \text{ such that } bw \in L \} \).
- \( \text{last}(L) = \{ b \mid \text{there is a word } w \text{ such that } wb \in L \} \).
- \( \text{follow}(L, a) = \{ b \mid \text{there are words } v \text{ and } w \text{ such that } vabw \in L \} \), for each symbol \( a \).
- \( \text{followlast}(L) = \{ b \mid \text{there are words } v \text{ and } w \text{ such that } v \in L, v \neq \varepsilon, \text{ and } vw \in L \} \).
Furthermore, we extend these sets to regular expressions $E$ by defining $\text{first}(E) = \text{first}(L(E))$ and similarly for the other sets.

**Lemma 2.1 [BE96].** For each marked regular expression $E$, a word $x_1 \cdots x_n$ over $\Pi$, $n \geq 1$, belongs to $L(E)$ if and only if the following three conditions hold:

1. $x_1 \in \text{first}(E)$.
2. $x_n \in \text{last}(E)$.
3. $x_{i+1} \in \text{follow}(E, x_i)$, for all $i$, $1 \leq i < n$.

The proof is a straightforward induction on $E$. It is essential, though, that $E$ is marked; for the regular expression $aa$, the word $aaa$ is not in $L(aa)$, although $a \in \text{first}(aa) \cap \text{last}(aa) \cap \text{follow}(aa, a)$.

Lemma 2.1 shows that, for each marked regular expression $E$, the conditions $u_1 z v_1, u_2 z v_2 \in L(E)$ imply $u_1 z v_2 \in L(E)$. Therefore, we can give an alternative characterization of 1-unambiguous regular expressions.

**Lemma 2.2.** A regular expression $E$ is 1-unambiguous if and only if the following two conditions hold:

1. For all $x, y$ in $\text{first}(E)$, $x \neq y$ implies $x^5 \neq y^5$.
2. For all $z \in \text{sym}(E)$ and $x, y$ in $\text{follow}(E', z)$, $x \neq y$ implies $x^5 \neq y^5$.

Glushkov [Glu61] and McNaughton and Yamada [MY60] were the first researchers to construct finite-state automata from marked regular expressions using the functions $\text{first}$, $\text{last}$, and $\text{follow}$; the automata are, however, deterministic. Motivated by the work of Glushkov, Book et al. [BEGO71] defined a nondeterministic automaton $G_E$, for each regular expression $E$, that we call the Glushkov automaton of $E$. Berstel and Pin [BE96] observed that an NFA for a regular expression $E$ can be constructed from the $\text{first}$, $\text{last}$, and $\text{follow}$ functions of $E$ as opposed to a marking of $E$, provided that the language denoted by $E$ is local. Berry and Sethi [BS86] showed that Glushkov automata are natural representations of regular expressions.

**Definition 2.3.** We define the Glushkov automaton $G_E = (Q_E, \Sigma, \delta_E, q_I, F_E)$ of a regular expression $E$ as follows:

1. $Q_E = \text{sym}(E') \cup \{q_I\}$; that is, the states of $G_E$ are the positions of $E'$ together with a new, initial state $q_I$.
2. For $a \in \Sigma$, $\delta_E(q_I, a) = \{x \mid x \in \text{first}(E'), x^5 = a\}$.
3. For $x \in \text{sym}(E')$ and $a \in \Sigma$, $\delta_E(x, a) = \{y \mid y \in \text{follow}(E', x), y^5 = a\}$.
4. $F_E = \{\text{last}(E') \cup \{q_I\}, \text{last}(E')\}$, if $\varepsilon \in L(E)$,
   otherwise.

**Proposition 2.3 [BEGO71].** $L(G_E) = L(E)$.

Since $L(E) = L(E')^2$, the proof is a direct consequence of Lemma 2.1. Some simple observations follow directly from the definition of the Glushkov automaton.
**Lemma 2.4.** Let $E$ be a regular expression. Then,

1. The Glushkov automaton $G_E$ has no transitions that lead to the initial state; that is, it is nonreturning \[\text{[Lei81]}\].

2. Any two transitions that lead to the same state in $G_E$ have identical labels.

Using Glushkov automata, we can give another characterization of 1-unambiguous regular expressions that is a direct consequence of Lemma 2.2.

**Lemma 2.5.** A regular expression $E$ is 1-unambiguous if and only if $G_E$ is deterministic; that is, if and only if $G_E$ is a DFA.

Figure 1a demonstrates that $(a+b)^* a + \varepsilon$ is not a 1-unambiguous regular expression. Nevertheless, the language denoted by $(a+b)^* a + \varepsilon$ is a 1-unambiguous language, since it is also denoted by $(b^*a)^*$, which is a 1-unambiguous regular expression; see Fig. 1b.

The Glushkov automaton $G_E$ can be computed in time quadratic in the size of $E$, which is worst-case optimal \[\text{[BK92a, BK93c, CP92, CP97]}\]. We can also construct the Glushkov automaton $G_E$ by induction on $E$. The inductive definition goes back to Mirkin \[\text{[Mir66]}\] and Leiss \[\text{[Lei81]}\]. Recently, Champarnaud \[\text{[Cha97]}\] showed that both methods produce the same automaton; see also Watson’s thesis \[\text{[Wat95]}\].

Book et al. \[\text{[BEGO71]}\] defined an NFA to be unambiguous if each word is denoted by at most one accepting path from the initial state to some final state. They also define unambiguity of regular expressions and prove that a regular expression is unambiguous if and only if its Glushkov automaton is unambiguous. Eilenberg \[\text{[Eil74]}\] introduced a notion of unambiguous finite-state automata and unambiguous regular languages that is different, yet related. He dealt with

![FIG. 1. The Glushkov automata corresponding to (a) $(a+b)^* a + \varepsilon$ and (b) $(b^*a)^*$.)](image-url)
generalized finite-state automata whose transitions have multiplicities; such a generalized finite-state automaton induces in a canonical way multiplicities on words that depend on the number of accepting paths in the automaton and on the multiplicities of the transitions. Then, an NFA $M$ is unambiguous in the sense of Book et al. if and only if the generalized finite-state automaton $M'$, whose transitions all have the multiplicity 1, denotes only words of multiplicity 1; that is, if and only if its language is unambiguous in the sense of Eilenberg.

3. DERIVATIVES OF 1-UNAMBIGUOUS LANGUAGES

We now prove that the family of 1-unambiguous languages is closed under derivatives [Brz64]. This result is essential for characterizing the 1-unambiguous languages. The proof makes use of a linear-time algorithm to convert regular expressions into star normal form [BK92a, BK93c]. We use the same technique, in Section 4, to obtain a Kleene theorem for 1-unambiguous languages.

Derivatives of languages and regular expressions were introduced by Brzozowski [Brz64], who used them to construct minimal DFAs for regular languages represented by regular expressions. We show that, given a 1-unambiguous regular expression $E$, its derivatives are also 1-unambiguous provided that $E$ is in star form. We also show that each 1-unambiguous language can be represented by a 1-unambiguous regular expression in star normal form. Thus, the family of 1-unambiguous languages is closed under derivatives.

**Definition 3.1.** The derivative $w \backslash L$ of a language $L$ with respect to a word $w$ is the language $\{v \mid wv \in L\}$.

**Definition 3.2.** The derivative $a \backslash E$ of a regular expression $E$ with respect to a symbol $a$ in $\Sigma$ is defined inductively as follows:

- $E = \emptyset$ or $E = \epsilon$: $a \backslash E = \emptyset$.
- $E = b$: $a \backslash E = \emptyset$, if $a = b$.
- $E = b$: $a \backslash E = \emptyset$, otherwise.
- $E = F + G$: $a \backslash E = a \backslash F + a \backslash G$.
- $E = FG$: $a \backslash E = (a \backslash F) G + a \backslash G$, if $\epsilon \in L(F)$,
  otherwise.
- $E = F^*$: $a \backslash E = (a \backslash F) F^*$.

**Proposition 3.1** [Brz64]. The family of regular languages is closed under derivatives. In particular, $L(a \backslash E) = a \backslash L(E)$, for each regular expression $E$.

**Example.** Let $H = ((ab)^* + c)^*$ and $I = (ab)^* + c$; then, $H = I^*$. Hence, $a \backslash H = (a \backslash I) I^*$ and

$$a \backslash I = a \backslash((ab)^* + c)$$
$$= (a \backslash(ab))(ab)^* + \emptyset$$
$$= ab(ab)^* + \emptyset.$$
Therefore,

\[ a \backslash H = ((ab)(ab)^* + \emptyset)((ab)^* + c)^*. \]

If we mark this regular expression as

\[ ((ab_1)(ab_2)^* + \emptyset)((ab_2)^* + c_4)^*, \]

we see that \( b_3 \) can be followed by \( a_2 \) and by \( a_4 \). Thus, \( a \backslash H \) is not 1-unambiguous, whereas \( H \) is 1-unambiguous.

It is the rule \( a \backslash E = (a \backslash F) F^* \) when \( E = F^* \) that gives derivatives that are not necessarily 1-unambiguous. In \( I \), we have a last position labeled \( a \) that can be followed by a first position labeled \( a \). In \( a \backslash I \), an \( a \) position can still follow a last position that is labeled \( b \). Considering \( a \backslash G = (a \backslash I) I^* \), however, the same \( b \) position in \( a \backslash I \) can now also be followed by a first position in \( I^* \) that is labeled \( a \); hence, \( a \backslash G \) is not 1-unambiguous. Expressions for which this situation cannot arise are said to be in star normal form \([BK92a, BK93c]\). For each regular expression \( E \) in star normal form, 1-unambiguity of \( E \) can be characterized solely in terms of the languages of the subexpressions of \( E \); see Lemma 3.2. Using this characterization, we can prove that derivatives of 1-unambiguous regular expressions in star normal form are again 1-unambiguous. As has already been mentioned, because each 1-unambiguous language can be denoted by a 1-unambiguous regular expression in star normal form, derivatives preserve 1-unambiguity of regular languages.

**Definition 3.3.** A regular expression \( E \) is in star normal form if, for each starred subexpression \( H \) of \( E \), followlast(\( H' \)) \( \cap \) first(\( H' \)) = \( \emptyset \) and \( \epsilon \notin L(\( H \)) \); in particular, no first position in \( H \) can follow a last position in \( H \).

**Lemma 3.2.** Let \( E \) be in star normal form.

- \( E = \emptyset, E = \epsilon, \text{ or } E = a; E \) is 1-unambiguous.
- \( E = F + G; E \) is 1-unambiguous if and only if \( F \) and \( G \) are 1-unambiguous and first(\( F \)) \( \cap \) first(\( G \)) = \( \emptyset \).
- \( E = FG; E \) is 1-unambiguous if and only if \( F \) and \( G \) are 1-unambiguous and first(\( F \)) \( \cap \) first(\( G \)) = \( \emptyset \).
- \( E = FG; E \) is 1-unambiguous if and only if \( F \) and \( G \) are 1-unambiguous and followlast(\( F \)) \( \cap \) first(\( G \)) = \( \emptyset \).
- \( E = F^*; E \) is 1-unambiguous if and only if \( F \) is 1-unambiguous and followlast(\( F \)) \( \cap \) first(\( F \)) = \( \emptyset \).

**Proof.** First note that we can prove a more general result by dropping the lemma-wide condition that \( E \) be in star normal form. If we do so, we should add an extra condition to the statement of the lemma (followlast(\( F' \)) \( \cap \) first(\( F' \)) = \( \emptyset \)) and change the proof of the last case slightly, as the we now have followlast(\( F' \)) \( \cap \) first(\( F' \)) = \( \emptyset \) by assumption.
We prove the three most interesting cases. First, we assume that \( E = FG \), \( L(E) \neq \emptyset \), and \( e \in L(F) \). Let \( F \) and \( G \) be 1-unambiguous, \( \text{first}(F) \cap \text{first}(G) = \emptyset \), and \( \text{followlast}(F) \cap \text{first}(G) = \emptyset \). We show that \( E \) is 1-unambiguous. Let \( E' = FG' \) and let \( ux, uyw \in L(E') \) such that \( x^2 = y^3 \). If \( x, y \in \text{sym}(F') \), then there are \( v_1 \) and \( v_1 \) such that \( xuv_1, yv_1 \in L(F') \), because \( E' \) is marked; hence, since \( F \) is 1-unambiguous, \( x = y \). The case when \( x, y \in \text{sym}(G') \) is analogous. Finally, we show that if \( x \in \text{sym}(F') \) and \( y \in \text{sym}(G') \) cannot occur. If \( u = e \), then \( x \in \text{sym}(F') \) and \( y \in \text{sym}(G') \) imply that \( x \in \text{first}(F') \) and \( y \in \text{first}(G') \); hence, \( x^2 \in \text{first}(F') \) and \( y^3 \in \text{first}(G') \), which contradicts the assumption that \( \text{first}(F) \cap \text{first}(G) = \emptyset \). If \( u \neq e \), then \( x \in \text{sym}(F') \) and \( y \in \text{sym}(G') \) imply that \( u \in L(F') \), because \( E' \) is marked; hence, \( x \in \text{followlast}(F') \) and \( y \in \text{first}(G') \). Therefore, \( x^2 \in \text{followlast}(F') \) and \( y^3 \in \text{first}(G') \), which contradicts the assumption that \( \text{followlast}(F) \cap \text{first}(G) = \emptyset \).

Second, let \( E = F^* \), \( F \) be 1-unambiguous, and \( \text{followlast}(F) \cap \text{first}(F) = \emptyset \). We show that \( E \) is 1-unambiguous. Let \( E' = F^* \). Lemma 2.1 implies that, for each word \( w \in L(E') \), there is exactly one canonical decomposition \( w = w_1 \cdots w_n \), \( n \geq 0 \), such that, for \( 1 \leq i \leq n \):

1. \( w_i \in L(F') \setminus \{e\} \); in particular, \( w_i \) starts with a symbol in \( \text{first}(F') \) and ends with a symbol in \( \text{last}(F') \).
2. \( w_i \) contains no subword \( xy \) such that \( x \in \text{last}(F') \) and \( y \in \text{first}(F') \).

Now let \( ux, uyw \in L(E') \) such that \( x^2 = y^3 \). If \( u = e \), then \( x, y \in \text{first}(F') \); hence, \( x = y \), because \( F \) is 1-unambiguous. So let us assume that \( u = u_0 z \). If \( z \in \text{last}(F') \) and \( x \in \text{first}(F') \), then \( y \in \text{first}(F') \) (otherwise \( y \in \text{followlast}(F') \) and \( y^3 \in \text{followlast}(F) \cap \text{first}(F') \)). Therefore, the canonical decompositions of \( ux \) and \( uyw \) separate either both of \( xz \) and \( zy \) or none of \( xz \) and \( zy \). Hence, \( ux \) and \( uyw \) can be decomposed as \( ux = u_1 \cdots u_n x v_1 \cdots v_l \) and \( uyw = u_1 \cdots u_n y w_1 \cdots w_k \) such that \( n \geq 1, k \geq 0, u_1, \ldots, u_n, v_1, \ldots, v_l, w_1, \ldots, w_k \in L(F') \). Because \( F \) is 1-unambiguous, \( x = y \).

Third, let \( E = F^* \) and \( a \in \text{followlast}(F) \cap \text{first}(F) \). We show that \( E \) is not 1-unambiguous. Once more let \( E' = F^* \). There are \( x, y, z, u, v, \) and \( w \) such that \( xu, vz, vzyw \in L(F') \) and \( x^2 = y^3 = a \). Therefore, \( x \in \text{first}(F') \) and \( y \in \text{followlast}(F) \); however, since \( E \) is in star normal form, \( \text{first}(F') \cap \text{followlast}(F) = \emptyset \), \( x \neq y \) and \( E \) is not 1-unambiguous.

**Theorem A.** Each 1-unambiguous language can be denoted by a 1-unambiguous regular expression in star normal form.

**Proof.** Given a regular expression \( E \), we can construct a regular expression \( E^* \) in star normal form such that \( G_{E^*} = G_E \) and \( L(E^*) = L(E) \) [BK92a, BK93c]. Hence, by Lemma 2.5, \( E^* \) is 1-unambiguous if and only if \( E \) is 1-unambiguous.

At this point, this paper is not quite self-contained; we have to refer to other papers [BK92a, BK93c] for a full proof of Theorem A. However, the main idea, why each regular expression can be transformed into star normal form without any changes in the Glushkov automaton, is quite easy to grasp. Given a regular expression \( H \), we can remove from its Glushkov automaton \( G_H \) all the transitions that lead from a state in \( \text{last}(H) \) to a state in \( \text{first}(H) \), and the resulting automaton is
the Glushkov automaton of another regular expression, say of $H$. Then, followlast($H^\ast$) $\cap$ first($H^\ast$) $=$ $\varnothing$ and $G_{H^\ast}$ $=$ $G_{H^\ast}$. Therefore, if we replace each subexpression $H^\ast$ of $E$ with $H^\ast$, the resulting regular expression $E^\ast$ is in star normal form and $G_{E^\ast}$ $=$ $G_{E^\ast}$. For example, if $H$ $=$ $a^*b^*$, then $H^\ast$ $=$ $a + b$; hence, the star normal form of $E$ $=$ $(a^*b^*)^\ast$ is $E^\ast$ $=$ $(a + b)^\ast$. Figure 2 demonstrates that the Glushkov automata of $(a^*b^*)^\ast$ and $(a + b)^\ast$ are identical.

The importance of star normal form for derivatives is captured in the following result.

**Theorem B.** If $E$ is a 1-ambiguous regular expression in star normal form, then $a \setminus E$ is 1-ambiguous and in star normal form, for each $a$ in $\Sigma$.

**Proof.** A straightforward induction on $E$ to show that $a \setminus E$ is in star normal form. Using Lemma 3.2, we show by induction on $E$ that $a \setminus E$ is 1-ambiguous.

$$E = \varnothing, E = \varepsilon, \text{ or } E = b; \ a \setminus E \text{ is 1-ambiguous.}$$

$$E = F + G; \text{ By the induction hypothesis, } a \setminus F \text{ and } a \setminus G \text{ are both 1-ambiguous. Furthermore, because first}(F) \cap \text{ first}(G) = \varnothing, \text{ we have } L(a \setminus F) = \varnothing \text{ or } L(a \setminus G) = \varnothing; \text{ hence, first}(a \setminus F) \cap \text{ first}(a \setminus G) = \varnothing. \text{ Thus, } a \setminus E \text{ is 1-ambiguous.}$$

$$E = FG; \text{ Without loss of generality, } L(a \setminus F) \neq \varnothing \text{ and } L(E) \neq \varnothing. \text{ Since } E \text{ is 1-ambiguous, } L(a \setminus F) \neq \varnothing \text{ implies } L(a \setminus G) = \varnothing. \text{ Therefore, we have only to show that } (a \setminus F) G \text{ is 1-ambiguous. By the induction hypothesis, } a \setminus F \text{ is 1-ambiguous. Furthermore, followlast}(a \setminus F) \cap \text{ first}(G) \text{ is a subset of followlast}(F) \cap \text{ first}(G); \text{ hence, it is empty. Finally, if } e \in L(a \setminus F), \text{ then first}(a \setminus F) \cap \text{ first}(G) \text{ is a subset of followlast}(F) \cap \text{ first}(G); \text{ hence, it is empty. Therefore, } (a \setminus F) G \text{ is 1-ambiguous.}$$

$$E = F^\ast; \text{ Since } E \text{ is 1-ambiguous and is in star normal form, followlast}(F) \cap \text{ first}(F) = \varnothing; \text{ thus, followlast}(a \setminus F) \cap \text{ first}(F^\ast) = \varnothing. \text{ Furthermore, if } e \in L(a \setminus F), \text{ then first}(a \setminus F) \cap \text{ first}(F^\ast) = \varnothing, \text{ because it is a subset of followlast}(F) \cap \text{ first}(F). \text{ Hence, by the induction hypothesis, } a \setminus E \text{ is 1-ambiguous.}$$

Because the derivative of a language $L$ with respect to a word is the repeated derivative of $L$ with respect to the word’s symbols, we need prove only that the derivative of $L$ with respect to a symbol is 1-ambiguous. Therefore, Theorems A and B immediately indicate the following result.

**Theorem C.** If $L$ is a 1-ambiguous language, then so is $w \setminus L$, for all $w$ in $\Sigma^\ast$.

**FIG. 2.** The Glushkov automaton corresponding to $(a^*b^*)^\ast$ and its star normal form $(a + b)^\ast$.
4. A KLEENE THEOREM FOR 1-UNAMBIGUOUS LANGUAGES

We characterize the family of 1-unambiguous languages in terms of closure properties. So far, we have seen one operation under which this family is closed: namely, derivatives. The family of 1-unambiguous languages is not closed under any of the operations that constitute the Kleene characterization of the regular languages, namely union, concatenation, and star. We can, however, define restrictions under which these operations preserve 1-unambiguous languages; the restricted operations are still powerful enough to generate each 1-unambiguous language from the usual primitives. The restrictions involve the first and last symbols of the words in a language.

It is convenient, for a language \( L \), to denote \( L \setminus \{e\} \) by \( L^- \).

**Theorem D.** The family of 1-unambiguous languages is the smallest family \( \mathcal{D} \) of languages that satisfies the following conditions:

1. \( \emptyset, e, \) and \( \{a\} \) are in \( \mathcal{D} \), for each symbol \( a \).
2. If \( A, B \in \mathcal{D} \) and first(\( A \)) \( \cap \) first(\( B \)) = \( \emptyset \), then \( A \cup B \in \mathcal{D} \).
3. If \( A, B \in \mathcal{D} \), \( e \notin A \), and followlast(\( A \)) \( \cap \) first(\( B \)) = \( \emptyset \), then \( AB \in \mathcal{D} \).
4. If \( A \in \mathcal{D} \), then \( A^- \in \mathcal{D} \).
5. If \( A \in \mathcal{D} \) and followlast(\( A \)) \( \cap \) first(\( A \)) = \( \emptyset \), then \( A^* \in \mathcal{D} \).

We prove Theorem D by first showing that \( L \) is 1-unambiguous if and only if \( L^- \) is 1-unambiguous. To this end, for each regular expression \( E \), we construct a regular expression \( E^- \) that denotes \( L(E)^- \). We then show that \( E \) is 1-unambiguous if and only if \( E^- \) is 1-unambiguous, provided that \( E \) is in star normal form.

**Definition 4.1.** For each regular expression \( E \), define \( E^- \) inductively as follows:

\[
E = \varepsilon \text{ or } E = \emptyset; E^- = \emptyset.
\]

\[
E = a; E^- = a.
\]

\[
E = F + G; E^- = F^- + G^-.
\]

\[
E = FG; E^- = \begin{cases} F^-G + G^- & \text{if } \varepsilon \in L(E), \\ FG & \text{otherwise.} \end{cases}
\]

\[
E = F^*; E^- = F^-F^*.
\]

**Lemma 4.1.**

1. \( L(E^-) = L(E)^- \).
2. If \( E \) is in star normal form, then so is \( E^- \).

The proof is a straightforward induction on \( E \).

**Lemma 4.2.** A regular expression \( E \) in star normal form is 1-unambiguous if and only if \( E^- \) is 1-unambiguous.

**Proof.** The proof is by induction on \( E \). Each subexpression of \( E \) and \( E^- \) is in star normal form. Thus, we can apply Lemma 3.2. We show only the case when...
E = FG and ε ∈ L(E). In this case, E^- = F^- G + G^-. If L(F^-) = ∅, then L(F) ⊆ {ε}; hence, both F and F^- are 1-unambiguous in this case. Now, E is 1-unambiguous if and only if F and G are 1-unambiguous, first(F) ∩ first(G) = ∅, and followlast(F) ∩ first(G) = ∅. On the other hand, by the induction hypothesis, E^- is 1-unambiguous if and only if F^- and G^- are 1-unambiguous, first(F^-) ∩ first(G) = ∅, and followlast(F^-) ∩ first(G) = ∅. This property holds even if L(F^-) = ∅ and Lemma 3.2 cannot be applied to F^- G. Again by the induction hypothesis, the two conditions are equivalent.

Proof of Theorem D. First, let 𝒟 be a family of languages that satisfies conditions 1 through 5. We show that 𝒟 contains all 1-unambiguous languages. For this purpose, we prove, by induction on E, that L(E) ∈ 𝒟 if E is in star normal form. It is a straightforward application of Lemma 3.2. We show only the case when E = FG, L(E) ≠ ∅, and ε ∈ L(F). By Lemma 3.2, F and G are 1-unambiguous, first(F) ∩ first(G) = ∅, and followlast(F) ∩ first(G) = ∅. Furthermore, L(E) = L(F^- L(G) ∪ L(G)). By the induction hypothesis, L(F^-) ∈ 𝒟 and L(G) ∈ 𝒟. Finally, ε ∈ L(F), followlast(L(F)) ∩ first(G) = followlast(F) ∩ first(G) = ∅, and first(L(F) ∩ first(G)) = first(L(G)) ∩ first(G) = ∅; hence, L(E) ∈ 𝒟.

To complete the proof we prove that the family of 1-unambiguous languages fulfills conditions 1 through 5. We show only that it fulfills condition 5. Let A be a 1-unambiguous regular expression such that followlast(A) ∩ first(A) = ∅. Without loss of generality, we assume that ε ∉ A. Let E be a 1-unambiguous regular expression in star normal form that denotes A. We show first that E* is also in star normal form; that is, that followlast(E*) ∩ first(E*) = ∅. It suffices to show that (followlast(E*) ∩ first(E*))^2 = ∅. This equation holds because (followlast(E*) ∩ first(E*))^2 is a subset of followlast(E*) ∩ first(E*), or, equivalently, of followlast(A) ∩ first(A). By Lemma 3.2, E* is 1-unambiguous. Thus, A*, which is denoted by E*, is a 1-unambiguous language.

5. THE RECOGNITION OF 1-UNAMBIGUOUS LANGUAGES

It is well known that, for each regular language L, the minimum-state deterministic automaton MS(L) of L is uniquely determined up to a renaming of states. We examine the structural properties of MS(L) that characterize a 1-unambiguous language L.

Starting from a regular expression E that denotes L, we can construct the minimal DFA MS(L) by applying the subset construction to the Glushkov automaton G_E, and then minimizing the resulting DFA using the equivalence-class construction [ASU86]. If E is a 1-unambiguous regular expression, however, then its Glushkov automaton is already a DFA and we do not need the subset construction. Thus, we look for structural properties of Glushkov automata that are preserved under minimization, but are not necessarily preserved under the subset construction.

Automata that contain no useless states are called trim [Eil74, Per90]. In this paper, we also consider G_∅, the Glushkov automaton of the empty set, to be trim.

A state is useless if it does not lie on any path from the initial state to some final state.
although its single state is useless. In this section, we denote regular languages by *trim regular expressions*; that is, regular expressions that are either syntactically identical to ∅ or do not contain ∅ as a syntactic constituent. On the one hand, it is not hard to see that the Glushkov automaton of a trim regular expression is also trim. On the other hand, ∅ in a regular expression E can introduce useless states into GE; consider, for example, a(∅b + c). As usual, we can transform a regular expression E into a trim regular expression E^t such that L(E) = L(E^t) by applying the following rules:

\[ E + ∅, ∅ + E \rightarrow E; E∅, ∅E \rightarrow ∅; ∅^* \rightarrow ε. \]

In the context of 1-unambiguous languages, this transformation is justified by the following lemma.

**Lemma 5.1.** Each 1-unambiguous language can be denoted by a trim, 1-unambiguous regular expression. In particular, a regular expression E is 1-unambiguous if and only if E^t is 1-unambiguous.

**Proof.** It is not hard to see that E^+ is a marking of E^t. Since L(E^t) = L(E^+), the regular expression E is 1-unambiguous if and only if E^t is 1-unambiguous.

Now we consider the cyclic structure of Glushkov automata that we describe in terms of orbits and gates. It turns out that the structure of orbits and gates is preserved under minimization; hence, orbits and gates are exactly the right tools for characterizing 1-unambiguous languages.

**Definition 5.1.** For a state q of an NFA M, the orbit of q, denoted by \( O(q) \), is the strongly connected component of q; that is, it is the set of states of M that can be reached from q and from which q can be reached. We consider the orbit of q to be trivial if it consists of only the state q and there are no transitions from q to itself in M.

**Definition 5.2.** A state q of an NFA M is a gate of its orbit \( O(q) \) if q is a final state or q has a transition to a state outside \( O(q) \). The NFA M has the orbit property if all the gates of each orbit have identical connections to the outside world. More precisely, if any pair \( q_1 \) and \( q_2 \) of gates in the same orbit satisfies the following two conditions:

1. \( q_1 \) is final if and only if \( q_2 \) is final.
2. For all states q outside the orbit of \( q_1 \) and \( q_2 \), there is a transition \((q_1, a, q)\) in M if and only if there is a transition \((q_2, a, q)\) in M.

The two automata in Fig. 3 have three orbits, namely \{1\}, \{2, 3\}, and \{4\}. The singleton orbits are trivial, and each state is a gate of its orbit. The orbit property holds for the automaton in Fig. 3a, whereas it does not hold for the one in Fig. 3b.

**Definition 5.3.** For a state q of an NFA M, let the orbit automaton \( M_q \) of q be the automaton obtained by restricting the state set of M to \( C(q) \) with initial state q and with the gates of \( C(q) \) as the final states of \( M_q \). The language of \( M_q \) is called the orbit language of q. The languages \( L(M_q) \) are also called the orbit languages.
of $M$. We also consider a larger subautomaton of $M$ related to $q$: The NFA $M'$ is $M$ with its state set restricted to the states reachable from $q$, with $q$ as its initial state, and its transitions are similarly restricted.

We now establish the following characterization of 1-unambiguous languages.

**Theorem E.** Let $M$ be a minimal DFA. Then, $L(M)$ is 1-unambiguous if and only if $M$ has the orbit property and all orbit languages of $M$ are 1-unambiguous. If $L(M)$ is 1-unambiguous, then a 1-unambiguous regular expression denoting $L(M)$ can be constructed from 1-unambiguous regular expressions for the orbit languages.

We begin the proof of Theorem E by demonstrating that it holds in the left-to-right direction. First, we show that the Glushkov automaton of a 1-unambiguous regular expression has the orbit property and that all its orbit languages are 1-unambiguous. Second, we show that these two properties are preserved under minimization. Thus, we first investigate the orbit structure of a Glushkov automaton. Then, we demonstrate how the orbit structure of a DFA is related to the orbit structure of the DFA’s equivalent minimal automaton.

Let us recall that, for each regular expression $E$, the state set of the Glushkov automaton $G_E$ consists of $\text{sym}(E)$, the true states of $G_E$, plus the initial state. Let $H$ be a subexpression of $E$. Then, $\text{sym}(H')$ is a subset of $\text{sym}(E')$. Given $x, y$ in $\text{sym}(H')$ such that $y \in \text{follow}(H', x)$, we also have $y \in \text{follow}(E', x)$, provided that $E$ is trim. Therefore, all transitions between true states of $G_H$ are also transitions in $G_E$. In this sense, $G_H$ is embedded in $G_E$. It is essential, though, that $E$ is trim, because otherwise $H$ can be blocked by the empty set in $E$; for example, when $E = aa\emptyset$ and $H = aa$, $G_H$ has two a transitions, but $G_E$ has none.

**Lemma 5.2.** Let $E$ be a trim regular expression and $x \in \text{sym}(E')$. Then, the orbits and gates in $G_E$ can be described as follows:

![Diagram](image-url)
1. If there is no starred subexpression \( H^* \) of \( E \) such that \( x \in \text{sym}(H) \), then \( \ell(x) = \{x\} \) and \( \ell(x) \) is trivial. On the other hand, if \( H^* \) is the maximal starred subexpression of \( E \) such that \( x \in \text{sym}(H) \), then \( \ell(x) = \text{sym}(H) \) and \( \ell(x) \) is not trivial. Finally, the orbit of the initial state is trivial.

2. If \( H^* \) is a maximal starred subexpression of \( E \) such that \( \text{sym}(H^*) \neq \emptyset \), then the last positions of \( H^* \) are the gates of the orbit given by \( \text{sym}(H^*) \). Furthermore, each transition of \( G_E \) between two states of \( \text{sym}(H^*) \) is already present in \( G_{H^*} \).

The proof is by a straightforward induction on \( E \).

**Lemma 5.3.** For each trim regular expression \( E \), the Glushkov automaton \( G_E \) satisfies the orbit property.

**Proof.** First, we observe that, for each subexpression \( H \) of \( E \), \( \text{last}(H') \subseteq \text{last}(E') \) or \( \text{last}(H') \cap \text{last}(E') = \emptyset \). Then, the claim follows from the previous lemma by induction on \( E \). We show only the case when \( E = FG \) and we consider only nontrivial orbits that are contained in \( \text{sym}(F) \). Thus, let \( H^* \) be a maximal starred subexpression of \( F \) such that \( \text{sym}(H^*) \neq \emptyset \). By the induction hypothesis, the orbit \( \text{sym}(H^*) \) of \( G_E \) satisfies the orbit property. If \( \text{last}(H') \cap \text{last}(E') = \emptyset \), then no gate of \( \text{sym}(H^*) \) is final in \( G_E \) and the transitions of \( G_E \) that leave the orbit \( \text{sym}(H^*) \) are the same as the transitions of \( G_F \) that leave \( \text{sym}(H') \). On the other hand, if \( \text{last}(H') \subseteq \text{last}(E') \), then all the gates of \( \text{sym}(H^*) \) in \( G_E \) are final states and, for each \( x \) in \( \text{first}(G_E) \), an \( x^2 \) transition to \( x \) is added to each gate of \( \text{sym}(H^*) \) in \( G_E \) in addition to the transitions that are already present in \( G_F \). Therefore, in both cases, the orbit given by \( \text{sym}(H^*) \) in \( G_E \) satisfies the orbit property.

**Lemma 5.4.** Let \( E \) be a trim, 1-unambiguous regular expression. Then, each orbit language of \( G_E \) is 1-unambiguous. In particular, if the orbit of state \( x \) in \( G_E \) is nontrivial, then there is a maximal starred subexpression \( H^* \) of \( E \) such that the orbit language of \( x \) is a derivative of \( L(H^*) \).

**Proof.** We consider only nontrivial orbits. Thus, let \( H^* \) be a maximal starred subexpression of \( E \) and \( x \in \text{sym}(H^*) \). Since \( L(H^*) \) is 1-unambiguous and the derivatives of 1-unambiguous languages are 1-unambiguous, it suffices to show that the orbit language \( L((G_E)_x) \) of \( x \) in \( G_E \) is a derivative of \( L(H^*) \). All transitions of \( G_E \) among the states of \( \text{sym}(H^*) \) are already transitions in \( G_{H^*} \). Furthermore, if we remove the initial state and its transitions from \( G_{H^*} \) and make \( x \) the new initial state, the resulting automaton is \((G_E)_x\). Now let \( w \) be a word that takes the initial state to \( x \) in \( G_{H^*} \). Because \( G_{H^*} \) is nonreturning and deterministic, \( L((G_E)_x) \) is the derivative of \( L(G_{H^*}) \) with respect to \( w \).

Now we consider the relationship between the orbit structure of a DFA and its minimization.

**Definition 5.4.** Let \( M \) be a DFA and \( \bar{M} \) be its equivalent minimal DFA. For a state \( q \) of \( M \) let \( [q] \) denote the set of states of \( M \) that are equivalent to \( q \). In particular, \( \{[q] \mid q \text{ state of } M\} \) is the set of states of \( \bar{M} \). Then, an orbit \( C \) of \( M \) is a lift of an orbit \( K \) of \( \bar{M} \) (or \( C \) lifts \( K \)) if \( K = \{[q] \mid q \in C\} \).
Lemma 5.5. Let $M$ be a DFA and let $\bar{M}$ be its equivalent minimal DFA. Then, for each orbit $K$ of $\bar{M}$ there is an orbit $C$ of $M$ such that

1. $C$ is a lift of $K$ and
2. for each $q$ in $C$, the orbit languages of $q$ in $M$ and $[q]$ in $\bar{M}$ are identical.

Proof. First we show that, given an orbit $K$ of $\bar{M}$ and a state $p$ of $M$ such that $[p] \in K$, there is an orbit $C$ of $M$ that is reachable from $p$ and lifts $K$. Indeed, there are states $q_0, q_1, \ldots, q_n$ of $M$, $p = q_0$, and symbols $a_0, \ldots, a_n$ such that

$[q_0] \xrightarrow{a_0} [q_1] \xrightarrow{a_1} \cdots \xrightarrow{a_n} [q_n] \xrightarrow{a_n} [q_0]
$

and

$K = \{[q_i] | 0 \leq i \leq n\}.$

Unfortunately, the states of $M$ reachable from $q_0$ with $a_0 \cdots a_n$, $0 \leq i \leq n$, do not necessarily belong to a single orbit of $M$. By repeatedly applying the same sequence $a_0 \cdots a_n$ to $q_0$ in $M$, however, we finally get into a cycle whose states all belong to a single orbit $C$ of $M$. By construction, $K \subseteq \{[q] | q \in C\}$. Furthermore, if $q$ and $r$ belong to one orbit of $M$, then $[q]$ and $[r]$ belong to one orbit in $\bar{M}$. Thus, $C$ lifts $K$.

The orbits of $M$ are partially ordered with respect to reachability; therefore, we can choose $C$ as a maximal (with respect to reachability) orbit $C'$ that lifts $K$. It suffices to prove the following two properties:

1. For each $q$ in $C$, $q$ is a gate of $C$ if and only if $[q]$ is a gate of $K$.
2. For all $q, r$ in $C$ and $a$ in $\Sigma$, if $[q] \xrightarrow{a} [r]$, then there is an $s$ in $C$ equivalent to $r$ such that $q \xrightarrow{a} s$.

The first claim follows from the second. So let $q, r \in C$, $[q] \xrightarrow{a} [r]$. Then, there is a state $s$ of $M$ equivalent to $r$ such that $q \xrightarrow{a} s$. We have to show that $s \in C$. Since $[s] = [r] \in K$, we can reach an orbit $C'$ of $M$ from $s$ that lifts $K$. The maximality of $C'$ implies $C = C'$; hence, $s \in C$.  

Lemma 5.6. Let $M$ be a DFA and $\bar{M}$ be its equivalent minimal DFA.

1. If $M$ satisfies the orbit property, then so does $\bar{M}$.
2. If all orbit languages of $M$ are 1-unambiguous, then so are all orbit languages of $\bar{M}$.

Proof. Let $K$ be an orbit of $\bar{M}$ and $C$ be a lift of $K$ according to Lemma 5.5. Furthermore, we assume that $M$ has an $a$ transition from $q$ in $C$ to $r$ outside $C$. We show that $[r] \notin K$.

Because $M$ is deterministic, $M_{\bar{a}}$ has no $a$ transition from $q$; hence, $a \setminus L(M_{\bar{a}}) = \emptyset$. Since the orbit languages of $q$ and $[q]$ are identical, $a \setminus L(M_{[q]}) = \emptyset$; hence, $M_{[q]}$ has no a transition from $[q]$. Yet, $M$ has an $a$ transition from $[q]$ to $[r]$; thus, $[r] \notin K$.

The proof of the lemma is completed by applying Lemma 5.5.
Proof of Theorem E. Lemmas 5.3, 5.4, and 5.6 establish the left-to-right direction. We show the implication from right to left by induction on the number of orbits of \( M \). We assume that \( M \) has more than one orbit and consider the orbit \( \mathcal{O}(q_I) \) of the initial state \( q_I \). Let \( a_1, \ldots, a_n \) be the distinct symbols of the transitions that leave \( \mathcal{O}(q_I) \). Since \( M \) satisfies the orbit property, there are states \( q_1, \ldots, q_n \) outside \( \mathcal{O}(q_I) \) such that all gates of \( \mathcal{O}(q_I) \) have an \( a_i \) transition to \( q_i \), and there are no other outgoing transitions from \( \mathcal{O}(q_I) \) to the outside. Since \( M \) is deterministic, \( M_{q_I} \) has no \( a_i \) transition from a final state.

We consider \( M^{o_1} \), the subautomaton of \( M \) whose states are the states of \( M \) that are reachable from \( q_I \) as the initial state. Because \( M^{o_1} \) is a minimal DFA that has fewer orbits than \( M \), the language of \( M^{o_1} \) is 1-unambiguous by the induction hypothesis. Furthermore, either

\[
L(M) = L(M_{q_I}) (a_1 L(M^{o_1}) \cup \cdots \cup a_n L(M^{o_n}))
\]
or

\[
L(M) = L(M_{q_I}) (a_1 L(M^{o_1}) \cup \cdots \cup a_n L(M^{o_n}) \cup \{z\}).
\]

By Lemma 3.2, a 1-unambiguous regular expression for \( M \) can be constructed from 1-unambiguous regular expressions for \( M_{q_I} \) and \( M^{o_1}, \ldots, M^{o_n} \), which completes the proof.

Theorem E suggests an inductive algorithm to determine, given a minimal DFA \( M \), whether \( L(M) \) is 1-unambiguous; in fact, if \( M \) satisfies the orbit property, then all orbit automata of \( M \) are also minimal. Yet we still have to cover the case when \( M \) consists of a single, nontrivial orbit. In this case, Lemmas 5.4 and 5.5 indicate that there is a 1-unambiguous regular expression \( H^* \) such that \( L(M) \) is a derivative of \( L(H^*) \). The question is, how can we construct \( G_H \) from \( M \)? Obviously, we have to cut from \( M \) the "feedback" transitions that distinguish \( G_H \) from \( G_H \). It turns out that, if we cut from \( M \) the maximum number of "feedback" transitions, then we arrive at an automaton \( N \) that consists of several disconnected subautomata. Each of these subautomata recognizes a 1-unambiguous language, and a 1-unambiguous regular expression for \( L(M) \) can be constructed from the 1-unambiguous regular expressions for the languages of the subautomata. Then, Theorem E can be applied to the subautomata of \( N \) recursively.

**Definition 5.5.** For a DFA \( M \), a symbol \( a \) in \( \Sigma \) is \( M \) consistent if there is a state \( f(a) \) in \( M \) such that all final states of \( M \) have an \( a \) transition to \( f(a) \). A set \( S \) of symbols is \( M \) consistent if each symbol in \( S \) is \( M \) consistent.

**Definition 5.6.** Let \( M \) be an NFA and \( S \) be a set of symbols. The \( S \) cut \( M_S \) of \( M \) is constructed from \( M \) by removing, for each \( a \in S \), all \( a \) transitions that leave a final state of \( M \).

Figure 4a gives a DFA for which \( \{a, b\} \) is consistent and Fig. 4b gives its \( \{a, b\} \) cut.

The following theorem is a generalization of Theorem E. It also deals with minimal automata that consist of a single, nontrivial orbit.
FIG. 4. DFAs and cuts: (a) A DFA for which \( \{a, b\} \) is consistent. (b) Its \( \{a, b\} \) cut.

**Theorem F.** Let \( M \) be a minimal DFA and \( S \) be an \( M \)-consistent set of symbols. Then, \( L(M) \) is 1-unambiguous if and only if

1. \( M_S \) satisfies the orbit property and
2. all orbit languages of \( M_S \) are 1-unambiguous.

Furthermore, if \( M \) consists of a single, nontrivial orbit and \( L(M) \) is 1-unambiguous, \( M \) has at least one \( M \)-consistent symbol.

We illustrate Theorem F with two examples. The language recognized by the automaton in Fig. 4a is 1-unambiguous, because its \( \{a, b\} \) cut has only trivial orbits; a 1-unambiguous regular expression for the whole language is \( c(a + b(c + cc))^* \).

Figure 5 shows the minimal DFA that recognizes \((0 + 1)^* 0(0 + 1)\). It consists of a single orbit with two gates, 00 and 01, but neither 0 nor 1 is consistent. Thus, \((0 + 1)^* 0(0 + 1)\) does not denote a 1-unambiguous language. Similarly, the regular expressions \((0 + 1)^* 0(0 + 1)^n\), for each \( n \geq 1 \), do not denote 1-unambiguous languages.

FIG. 5. The minimal DFA for \((0 + 1)^* 0(0 + 1)\).
We need some preparation before we can prove Theorem F. The relationship between a DFA and its cuts is captured in the following result.

**Lemma 5.7.** Let $M$ be a trim DFA and $S$ be an $M$-consistent set of symbols. Then, $L(M)$ is 1-unambiguous if and only if, for each state $q$, the language $L(M^q)$ is 1-unambiguous. If $L(M)$ is 1-unambiguous, a 1-unambiguous regular expression denoting $L(M)$ can be constructed from 1-unambiguous regular expressions for the languages $L(M^q)$.

**Proof.** First, we show that

$$L(M) = L(M_S) \left( \bigcup_{a \in S} aL(M^{(a)}) \right)^*.$$  

Let $w = a_1 \cdots a_n \in L(M)$, $n \geq 1$, let

$$s = \{ i \mid 1 \leq i \leq n, a_i \in S, a_1 \cdots a_{i-1} \in L(M) \} \cup \{ n+1 \},$$

and let $i_1, \ldots, i_k$ be the elements of $s$ in increasing order. Then,

1. $a_1 \cdots a_{i_{k-1}} \in L(M_S)$ and
2. $a_{i_k} \cdots a_{i_{k+1}} \in a_{i_k}L(M^{(a_{i_k})}),$

for $1 \leq j < k$, because $M$ is deterministic. Thus,

$$w \in L(M_S) \left( \bigcup_{a \in S} aL(M^{(a)}) \right)^*.$$  

On the other hand, if $w \in L(M_S), a_i \in S, w_i \in L(M^{(a_i)})$, for $1 \leq i \leq n$, $n \geq 0$, then $w$ takes the initial state to a final state in $M$, $a_i$ takes the final state to $f(a_i)$ in $M$, and $a_i$ takes $f(a_i)$ to a final state in $M$, for $1 \leq i \leq n$. Thus, $wa_1w_1 \cdots a_nw_n \in L(M)$.

Next, $\text{followlast}(L(M_S)) \subseteq \Sigma \backslash S$ and $\text{followlast}(aL(M^{(a)})) \subseteq \Sigma \backslash S$. Furthermore, if $e \in L(M_S)$, then $e \in L(M)$; hence, $\text{first}(L(M^q)) \subseteq \Sigma \backslash S$. By Lemma 3.2, if $L(M^q)$ is 1-unambiguous, for each state $q$ of $M$, then $L(M)$ is 1-unambiguous.

Finally, we assume that $L(M)$ is 1-unambiguous; that is, there is a trim, 1-unambiguous regular expression $E$ in star normal form that denotes $L(M)$. Since $M$ is trim, there is, for each state $q$, a word $w$ that takes the initial state to $q$ in $M$. Let $p$ be the state to which $w$ takes the initial state in $G_E$. Since $M$ and $G_E$ are both deterministic, $L(M^q) = L(G_E^q)$. By Lemma 5.8, which we prove later, $L(G_E^q)$ is 1-unambiguous.

In the proof of Lemma 5.7, we have used the fact that $L(G_E)$ is 1-unambiguous, for each trim, 1-unambiguous regular expression $E$ in star normal form. To prove this fact, we define a regular expression $E^S$ that denotes $L(G_E)$ and show that $E^S$ is 1-unambiguous and in star normal form if $E$ is. For $I = \{ i_1, \ldots, i_n \}$, $n \geq 0$, we use $\bigoplus_{i \in I} E_i$ as shorthand for the regular expression $E_{i_1} + \cdots + E_{i_n}$.

**Definition 5.7.** Let $E$ be a trim regular expression, $S$ be a set of symbols, and $q$ be a state of $G_E$; that is, $q = q_f$ or $q \in \text{sym}(E)$. We define a regular expression $E^S$ that denotes $L(G_E^q)$ by induction on $E$ as follows:
\[ E = \emptyset \] or \[ E = e : E^a_3 = E. \]

\[ E = a : E^a_3 = \begin{cases} a, & \text{if } q = q_1, \\ e, & \text{if } q \notin \text{sym}(E') \end{cases} \]

\[ E = F + G : E^a_3 = \begin{cases} F^a_3 + G^a_3, & \text{if } q = q_1, \ e \in \text{L}(E), \\ F^a_3, & \text{if } q \in \text{sym}(F'), \\ G^a_3, & \text{if } q \in \text{sym}(G'). \end{cases} \]

\[ E = FG : E^a_3 = \begin{cases} F^a_3 \bigoplus_{a \notin S} (aa\backslash F'_3), & \text{if } q = q_1 \text{ or } q \in \text{sym}(F'), \ e \in \text{L}(G). \\ F^a_3, & \text{if } q = q_1 \text{ or } q \in \text{sym}(F'), \ e \notin \text{L}(G), \\ G^a_3, & \text{if } q \in \text{sym}(G'). \end{cases} \]

\[ E = F^* : E^a_3 = F^a_3 \bigoplus_{a \notin S} (aa\backslash F'_3)^*. \]

It is a straightforward, though laborious, induction to show that \( \text{L}(G_{E_3}) = \text{L}(E_3^a) \), for each trim regular expression \( E \).

**Lemma 5.8.** Let \( E \) be a trim regular expression, \( S \) be a set of symbols, and \( q \) be a state of \( G_E \). If \( E \) is \( 1 \)-unambiguous and in star normal form, then so is \( E^a_3 \).

**Proof.** The proof is by induction on \( E \). We apply the inductive definition of \( 1 \)-unambiguity given in Lemma 3.2. Furthermore, we make use of the fact that \( \text{first}(aa\backslash H) \subseteq \text{first}(H) \), because, if \( a \notin \text{first}(H) \), then \( \text{L}(aa\backslash H) = \emptyset \) and, thus, \( \text{first}(aa\backslash H) = \emptyset \).

We establish the three most interesting cases. First, let \( E = F + G, q = q_1, \) and \( e \in \text{L}(E) \). By the induction hypothesis, \( F^a_3 \) and \( G^a_3 \) are \( 1 \)-unambiguous and in star normal form; hence, by Theorem B, so are \( a\backslash F^a_3 \) and \( a\backslash G^a_3 \). Furthermore, \( \text{first}(aa\backslash F^a_3) \cap \text{first}(aa\backslash G^a_3) \) is a subset of \( \text{first}(F^a_3) \cap \text{first}(G^a_3) \) and, thus, of \( \text{first}(F) \cap \text{first}(G) \), which is empty. Therefore, \( E^a_3 \) is also \( 1 \)-unambiguous and in star normal form.

Second, let \( E = FG, q = q_1, \) or \( q \in \text{sym}(F') \), and \( e \notin \text{L}(G) \). Then, \( \text{followlast}(F^3_g) \subseteq \text{followlast}(F') \); hence, \( \text{followlast}(F^3_g) \cap \text{first}(G^a_3) \subseteq \text{followlast}(F) \cap \text{first}(G) = \emptyset \). Furthermore, if \( e \in \text{L}(F^3_g) \) and \( q = q_1 \), then \( e \in \text{L}(F) \) and \( \text{first}(F^3_g) = \text{first}(F) \); hence, \( \text{first}(F^3_g) \cap \text{first}(G^a_3) \subseteq \text{first}(F) \cap \text{first}(G) = \emptyset \). On the other hand, if \( e \in \text{L}(F^3_g) \) and \( q \notin \text{sym}(F') \), then \( q \in \text{last}(F) \), because \( F \) is trim; hence, \( \text{first}(F^3_g) \cap \text{first}(G^a_3) \subseteq \text{followlast}(F) \cap \text{first}(G) = \emptyset \). Finally, by the induction hypothesis, \( F^a_3 \) and \( G^a_3 \) are \( 1 \)-unambiguous and in star normal form. Thus, \( E^a_3 \) is also \( 1 \)-unambiguous and in star normal form.

Third, let \( E = F^* \). Then, \( \text{followlast}(F^a_3) \cap \text{first}(aa\backslash F'_3) \subseteq \text{followlast}(F) \cap \text{first}(F) = \emptyset \), since \( E \) is in star normal form. Next, if \( e \in \text{L}(F^a_3) \), then \( q \neq q_1 \), because \( E \) is in star normal form implies that \( e \notin \text{L}(F) \); hence, as in the previous case, \( \text{first}(F^a_3) \cap \text{first}(aa\backslash F'_3) \subseteq \text{followlast}(F) \cap \text{first}(F) = \emptyset \). Finally, \( \text{followlast}(aa\backslash F'_3) \subseteq \text{followlast}(F'); \) hence, for \( H = \bigoplus_{a \in S} (aa\backslash F'_3), \text{followlast}(H) \cap \text{first}(H) \subseteq \text{followlast}(F') \cap \text{first}(F) = \emptyset \). Therefore, \( \text{followlast}(H') \cap \text{first}(H') = \emptyset \). By the induction hypothesis and Theorem B, \( E^a_3 \) is \( 1 \)-unambiguous and in star normal form.
To prove Theorem F, we need to apply Lemma 5.7 and Theorem E. We now prove that the preconditions of Theorem E are satisfied.

**Lemma 5.9.** Let $M$ be a minimal DFA and $S$ be an $M$-consistent set of symbols. Then, $M_S^q$ is minimal, for each state $q$ of $M$.

**Proof.** We show that no two states of $M_S$ are equivalent in $M_S$. Let $p$ and $q$ be two distinct states of $M$ and $w$ be a word of minimal length that distinguishes between $p$ and $q$; that is, that takes one, but not both, of $p$ and $q$ to a final state in $M$. If $w = uav$ and $a \in S$, then the minimality of $w$ implies that $u$ takes neither $p$ nor $q$ to a final state in $M$, or $u$ takes both $p$ and $q$ to a final state in $M$. However, if $u$ takes $p$ and $q$ to a final state in $M$, then $ua$ takes $p$ and $q$ to the same state $f(a)$, because $a$ is $M$ consistent; thus, in this case, $w$ cannot distinguish between $p$ and $q$. Therefore, $u$ takes neither $p$ nor $q$ to a final state in $M$; hence, $w$ distinguishes between $p$ and $q$ in $M_S$.

**Proof of Theorem F.** Let $M$ be a minimal DFA and $S$ be an $M$-consistent set of symbols. By Lemma 5.9, $M_S^q$ is minimal, for each state $q$ of $M$. By Lemma 5.7 and Theorem E, $L(M)$ is 1-unambiguous if and only if each state $q$ of $M$ satisfies the following two conditions:

1. $M_S^q$ satisfies the orbit property and is trim.
2. Each orbit language of $M_S^q$ is 1-unambiguous.

However, the orbit automaton of $q$ in $M_S^q$ is identical to the orbit automaton of $q$ in $M_S$ and the transitions that leave the orbit of $q$ in $M_S^q$ are the same as the transitions that leave the orbit of $q$ in $M_S$. Therefore, $L(M)$ is 1-unambiguous if and only if the two conditions in the theorem hold.

Now let $M$ consist of a single, nontrivial orbit and $L(M)$ be 1-unambiguous. Lemmas 5.4 and 5.5 imply that there is a 1-unambiguous regular expression $E^*$ such that $L(M) = L(G_{E^*}^M)$, for some state $q$. Furthermore, $first(E)$ is $G_{E^*}^M$ consistent. Since $M$ is the minimization of $G_{E^*}^M$, $first(E)$ is also $M$ consistent. Because $M$ is a nontrivial orbit, $first(E) \neq \emptyset$, which concludes the proof.

**Theorem F** gives rise to the following decidability result for 1-unambiguous regular languages, which we prove after establishing one preliminary result.

**Theorem G.** It is decidable whether the language of a DFA $M$ is 1-unambiguous and the decision algorithm runs in time quadratic in the size of $M$. If the language of the DFA $M$ is 1-unambiguous, then an equivalent 1-unambiguous regular expression can be constructed in time exponential in the size of $M$.

Starting from a minimal DFA $M$ and a set $S$ of $M$-consistent symbols, we apply Theorem F recursively to the orbit automata of $M_S$. For this reason we first ensure that the orbit automata are also minimal.
Lemma 5.10. Let \( M \) be a minimal DFA that satisfies the orbit property. Then, \( M_q \) is minimal, for each state \( q \) of \( M \).

Proof. Let \( w \) be a word in \( \Sigma^* \) that distinguishes between two states \( p \) and \( q \) in \( M \) that belong to the same orbit of \( M \). Then, \( w \) takes one, but not both of \( p \) and \( q \) to a final state in \( M \).

Let \( u \) be the longest subword of \( w \) such that both computations with \( u \) starting from \( p \) and from \( q \) stay within \( M_q \). If \( u = w \), then \( w \) distinguishes between \( p \) and \( q \) in \( M_q \). Otherwise, consider the symbol \( a \) in \( \Sigma \) such that \( ua \) is a subword of \( w \). We assume without loss of generality that the computation of \( ua \) starting from \( p \) does not stay within \( M_q \). There can be two reasons for this occurrence. The first possibility is that \( u \) takes \( p \) to a final state in \( M_q \) and there is an \( a \) transition from the final state out of the orbit of \( q \). Then, \( u \) does not take \( q \) to a final state in \( M_q \), because otherwise \( ua \) would take \( p \) and \( q \) to the same state in \( M \) (by the orbit property). Hence, \( w \) cannot distinguish between \( p \) and \( q \) in \( M \). Therefore, \( u \) distinguishes between \( p \) and \( q \) in \( M_q \). The second possibility is that \( u \) takes \( p \) to a state in \( M_q \) that has no \( a \) transition in \( M \). Then, \( u \) takes \( q \) to a final state in \( M_q \). Thus, there is a subword \( v \) of \( w \) that is at least as long as \( u \) and takes \( q \) to a final state in \( M_q \). This word \( v \) distinguishes \( p \) from \( q \) in \( M_q \).

Proof of Theorem G. Since the minimal DFA for each DFA \( M \) can be computed in time quadratic in the size of \( M \) [HU79], we can assume without loss of generality that \( M \) is minimal. Figure 6 defines a function \( 1\text{-unambiguous}(M) \) that returns true if and only if \( L(M) \) is 1-unambiguous.

After statement 5 of Fig. 6, each orbit automaton of \( M_S \) is smaller than \( M \) and is minimal, since \( M_S \) has the orbit property. Therefore, the algorithm halts.

If \( x \) and \( y \) belong to the same orbit of \( M_S \), then \( L((M_S)x) \) is a derivative of \( L((M_S)y) \) and vice versa; hence, \( L((M_S)x) \) is 1-unambiguous if and only if \( L((M_S)y) \) is 1-unambiguous. Therefore, by Theorem F, the algorithm is correct.

Because the strongly connected components of a directed graph can be computed in linear time [AHU74], Steps 1 through 5 of the algorithm take time linear in the

function 1-unambiguous\( (M : \text{minimal DFA}) : \text{Boolean} \);  
1 compute \( S := \{ a \in \Sigma \mid a \text{ is } M \text{ consistent} \} \);  
2 if \( M \) has a single, trivial orbit then return true  
3 if \( M \) has a single, nontrivial orbit and \( S = \emptyset \) then return false;  
4 compute the orbits of \( M_S \);  
5 if not OrbitProperty\( (M_S) \) then return false;  
6 for each orbit \( K \) of \( M_S \) do  
   choose \( x \) in \( K \);  
   if not 1-unambiguous\( ((M_S)x) \) then return false;  
end;  
7 return true;

FIG. 6. The decision algorithm for 1-unambiguity of languages denoted by minimal DFAs.
6. CONCLUSIONS

The definition of 1-unambiguity in the SGML standard gives rise to at least two questions, both of theoretical interest and of relevance for systems that support SGML.

1. Is it decidable, given a model group $E$, whether $E$ is unambiguous? And, if it is, then what is its time complexity?

2. Is it decidable, given a regular language $L$ (represented, for example, by a finite-state automaton), whether $L$ can be represented by an unambiguous model group? And, if it is, then what is its time complexity?

We have demonstrated in Section 2 the decidability of the first question for standard regular expressions; the implicit algorithm has a time complexity that is quadratic in the size of the regular expression. Brüggemann-Klein [BK92a,
BK93c] designed a linear-time algorithm for standard regular expressions and Ponty et al. [PZD97] showed how to represent the Glushkov automaton in linear space. For model groups, Brüggenmann-Klein [BK93a, BK93b] demonstrated decidability.

We have presented a solution for the second question for standard regular expressions. We leave as an open problem, the characterization of the family of languages that can be denoted by 1-unambiguous model groups. A further research topic is to investigate $k$-unambiguous regular languages and regular expressions, when the lookahead is a constant $k \geq 1$.

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ONE-UNAMBIGUOUS REGULAR LANGUAGES


