Agent-Based Systems Tutorial 6

Version with suggested solutions

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Suggestions for solutions are printed in italics below each question

Q1 Consider the politics in the UK example from the lecture: $\Omega = \{\omega_L, \omega_D, \omega_C\}$, where ω_L represents the Labour Party, ω_D the Liberal Democrats and ω_C the Conservative Party. Voters have the following preferences:

- 43% of $|Ag|$ are left-wing voters: $\omega_L \succ \omega_D \succ \omega_C$
- 12% of $|Ag|$ are centre-left voters: $\omega_D \succ \omega_L \succ \omega_C$
- 45% of $|Ag|$ are right-wing voters: $\omega_C \succ \omega_D \succ \omega_L$

1. Which party will win an election based on the following voting procedures:
   - Plurality
   - Sequential majority elections with $\omega_L, \omega_D, \omega_C$
2. Is it possible to fix the election agenda in favour of any outcome?
3. Assuming that a new fourth party ω_N emerges altering the preferences of the voters to:
   - 38% of $|Ag|$ are left-wing voters: $\omega_L \succ \omega_D \succ \omega_N \succ \omega_C$
   - 11% of $|Ag|$ are centre-left voters: $\omega_D \succ \omega_L \succ \omega_N \succ \omega_C$
   - 39% of $|Ag|$ are right-wing voters: $\omega_C \succ \omega_D \succ \omega_L \succ \omega_N$
   - 12% of $|Ag|$ are voters of the new party: $\omega_N \succ \omega_C \succ \omega_D \succ \omega_L$

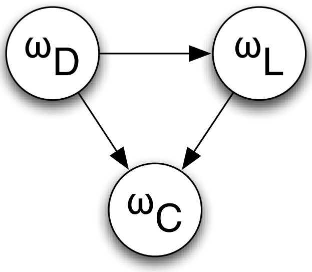
   In favour of which party is it possible to fix the election agenda in sequential majority elections?
4. Determine the winner of the election using the following voting procedures:
   - The Borda count
   - The Slater ranking

Solution suggestions:

1. Plurality: The outcome that appears first in most preference orders wins. Winner is ω_C
   Sequential majority elections are based on multiple pairwise elections. The order of the elections is defined by the election agenda. The election agenda $\omega_L, \omega_D, \omega_C$ denotes that ω_L will initially face ω_D. Then, the winner will go on to face ω_C, determining the winner of the overall election.

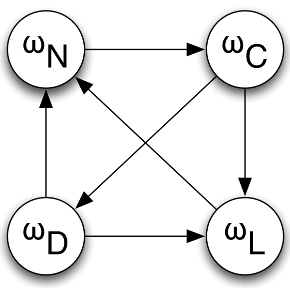
   - ω_D wins against ω_L, since 57% of the voters rank it higher than ω_L.
   - ω_D wins against ω_C, since 55% of the voters rank it higher than ω_C.
   - ω_D wins the election.

2. We construct a majority graph. Nodes in the graph correspond to outcomes, i.e. $\omega_D, \omega_L, \omega_C$. There is an edge from ω to ω' if a majority of voters rank ω above ω'. From the preferences of the voters we have:
   - ω_D wins against ω_L, since 57% of the voters rank it higher than ω_L.
   - ω_D wins against ω_C, since 55% of the voters rank it higher than ω_C.
   - ω_L wins against ω_C, since 55% of the voters rank it higher than ω_C.

   We are able to fix the election in favour of an outcome if there exists some agenda that would result in this outcome being the overall winner. Then the outcome is called a possible winner. We can check if an outcome ω_i is a possible winner if there exists a path from ω_i to every other node ω_j in the majority graph. The outcome ω_D is the only possible winner. Also, since there is an edge from ω_D to every other node in the graph, ω_D is the Condorcet winner, i.e. the overall winner for every possible agenda.

3. We construct the majority graph for the updated problem:
   - ω_D wins against ω_L, since 62% of the voters rank it higher.
   - ω_C wins against ω_D, since 51% of the voters rank it higher.
   - ω_C wins against ω_L, since 51% of the voters rank it higher.
   - ω_L wins against ω_N, since 88% of the voters rank it higher.
   - ω_D wins against ω_N, since 88% of the voters rank it higher.
   - ω_N wins against ω_C, since 61% of the voters rank it higher.

   There is a path from every node to every other node. Therefore, all outcomes are possible winners. We can use the following agendas to fix the elections:

  - ω_L wins with the agenda $\omega_D, \omega_C, \omega_N, \omega_L$.
  - ω_D wins with the agenda $\omega_C, \omega_N, \omega_L, \omega_D$.
  - ω_C wins with the agenda $\omega_N, \omega_L, \omega_D, \omega_C$.
  - ω_N wins with the agenda $\omega_L, \omega_D, \omega_C, \omega_N$.

4. The Borda count looks at the entire preference ordering, counts the strength of opinion in favour of a candidate. In order to calculate it: for all preference orders and outcomes ($|\Omega = k|$), if ω_i is lth in a preference ordering, increment its strength by $k - l$. For simplicity we calculate the Borda count for 100 voters.

$$\omega_L: \; 38 * (4-1) + 11 * (4-2) + 39 * (4-3) + 12 * (4-4) = 114 + 22 + 39 = 175$$
$$\omega_D: \; 38 * (4-2) + 11 * (4-1) + 39 * (4-2) + 12 * (4-3) = 76 + 33 + 78 + 12 = 199$$
$$\omega_C: \; 38 * (4-4) + 11 * (4-4) + 39 * (4-1) + 12 * (4-2) = 117 + 24 = 141$$
$$\omega_N: \; 38 * (4-3) + 11 * (4-3) + 39 * (4-4) + 12 * (4-1) = 38 + 11 + 36 = 85$$

  - Therefore, the winner is ω_D

The Slater rankning tries to minimise the disagreements between the majority graph and the social choice. We need to measure the degree of disagreement for each possible ordering using the majority graph. The degree of disagreement is the the number of edges that need to be flipped to make the ordering consistent with the majority graph.

$$\begin{array}{lccl}
\omega_D \succ^* \omega_C \succ^* \omega_L \succ^* \omega_N: & 2 & : & (\omega_D, \omega_C)\ (\omega_C, \omega_N) \\
\omega_D \succ^* \omega_C \succ^* \omega_N \succ^* \omega_L: & 3 & : & (\omega_D, \omega_C)\ (\omega_C, \omega_N)\ (\omega_N, \omega_L) \\
\omega_D \succ^* \omega_L \succ^* \omega_C \succ^* \omega_N: & 3 & : & (\omega_D, \omega_C)\ (\omega_L, \omega_C)\ (\omega_C, \omega_N) \\
\omega_D \succ^* \omega_L \succ^* \omega_N \succ^* \omega_C: & 2 & : & (\omega_D, \omega_C)\ (\omega_L, \omega_C) \\
\omega_D \succ^* \omega_N \succ^* \omega_C \succ^* \omega_L: & 2 & : & (\omega_D, \omega_C)\ (\omega_N, \omega_L) \\
\omega_D \succ^* \omega_N \succ^* \omega_L \succ^* \omega_C: & 3 & : & (\omega_D, \omega_C)\ (\omega_N, \omega_L)\ (\omega_L, \omega_C) \\
\omega_C \succ^* \omega_D \succ^* \omega_L \succ^* \omega_N: & 1 & : & (\omega_C, \omega_N) \\
\omega_C \succ^* \omega_D \succ^* \omega_N \succ^* \omega_L: & 2 & : & (\omega_C, \omega_N)\ (\omega_N, \omega_L) \\
\omega_C \succ^* \omega_L \succ^* \omega_D \succ^* \omega_N: & 2 & : & (\omega_C, \omega_N)\ (\omega_L, \omega_D) \\
\omega_C \succ^* \omega_L \succ^* \omega_N \succ^* \omega_D: & 3 & : & (\omega_C, \omega_N)\ (\omega_L, \omega_D)\ (\omega_N, \omega_D) \\
\omega_C \succ^* \omega_N \succ^* \omega_D \succ^* \omega_L: & 3 & : & (\omega_C, \omega_N)\ (\omega_N, \omega_D)\ (\omega_N, \omega_L) \\
\omega_C \succ^* \omega_N \succ^* \omega_L \succ^* \omega_D: & 4 & : & (\omega_C, \omega_N)\ (\omega_N, \omega_L)\ (\omega_N, \omega_D)\ (\omega_L, \omega_D) \\
\omega_L \succ^* \omega_D \succ^* \omega_C \succ^* \omega_N: & 4 & : & (\omega_L, \omega_D)\ (\omega_L, \omega_C)\ (\omega_D, \omega_C)\ (\omega_C, \omega_N) \\
\omega_L \succ^* \omega_D \succ^* \omega_N \succ^* \omega_C: & 3 & : & (\omega_L, \omega_D)\ (\omega_L, \omega_C)\ (\omega_D, \omega_C) \\
\omega_L \succ^* \omega_C \succ^* \omega_D \succ^* \omega_N: & 3 & : & (\omega_L, \omega_C)\ (\omega_L, \omega_D)\ (\omega_C, \omega_N) \\
\omega_L \succ^* \omega_C \succ^* \omega_N \succ^* \omega_D: & 4 & : & (\omega_L, \omega_C)\ (\omega_L, \omega_D)\ (\omega_C, \omega_N)\ (\omega_N, \omega_D) \\
\omega_L \succ^* \omega_N \succ^* \omega_D \succ^* \omega_C: & 4 & : & (\omega_L, \omega_D)\ (\omega_L, \omega_C)\ (\omega_N, \omega_D)\ (\omega_D, \omega_C) \\
\omega_L \succ^* \omega_N \succ^* \omega_C \succ^* \omega_D: & 3 & : & (\omega_L, \omega_C)\ (\omega_L, \omega_D)\ (\omega_N, \omega_D) \\
\omega_N \succ^* \omega_D \succ^* \omega_C \succ^* \omega_L: & 3 & : & (\omega_N, \omega_D)\ (\omega_N, \omega_L)\ (\omega_D, \omega_C) \\
\omega_N \succ^* \omega_D \succ^* \omega_L \succ^* \omega_C: & 4 & : & (\omega_N, \omega_D)\ (\omega_N, \omega_L)\ (\omega_D, \omega_C)\ (\omega_L, \omega_C) \\
\omega_N \succ^* \omega_C \succ^* \omega_D \succ^* \omega_L: & 2 & : & (\omega_N, \omega_D)\ (\omega_N, \omega_L) \\
\omega_N \succ^* \omega_C \succ^* \omega_L \succ^* \omega_D: & 3 & : & (\omega_N, \omega_L)\ (\omega_N, \omega_D)\ (\omega_L, \omega_D) \\
\omega_N \succ^* \omega_L \succ^* \omega_D \succ^* \omega_C: & 5 & : & (\omega_N, \omega_L)\ (\omega_N, \omega_D)\ (\omega_L, \omega_D)\ (\omega_L, \omega_C)\ (\omega_D, \omega_C) \\
\omega_N \succ^* \omega_L \succ^* \omega_C \succ^* \omega_D: & 4 & : & (\omega_N, \omega_L)\ (\omega_N, \omega_D)\ (\omega_L, \omega_C)\ (\omega_L, \omega_D)
\end{array}$$

We now select the ordering with the lower cost, which is $\omega_C \succ^* \omega_D \succ^* \omega_L \succ^* \omega_N$.

Q2 Consider the following coalitional games:

- (The glove game) Players have left and right hand gloves and they are trying to form pairs. Players 1 and 2 have right hand gloves whereas player 3 has a left hand glove. The agents have the following value function:

$$v(C) = \begin{cases} 1 & \text{if } C \in \{\{1,3\},\{2,3\},\{1,2,3\}\} \\ 0 & \text{otherwise} \end{cases}$$

- (The treasure of Sierra Madre game) 3 people find a treasure of many gold pieces in the mountains of Sierra Madre. Each piece can be carried by two people but not by a single person. The valuation function of this game is:

$$v(C) = \lfloor \frac{|C|}{2} \rfloor$$

1. Compute the Core
2. Compute the Shapley value for both games.

Solution suggestions:

1. We start by describing the Core. An outcome $x = \langle x_1, \ldots, x_k \rangle$ for a coalition C in game $\langle Ag, v \rangle$ is a distribution of C's utility to members of C. Outcomes must be feasible (don't overspend) and efficient (don't underspend): $\sum_{i \in C} x_i = v(C)$. C objects to an outcome for the grand coalition if there is some outcome for C in which all members of C are strictly better off. Formally, $C \subseteq Ag$ objects to $x = \langle x_1, \ldots, x_n \rangle$ for the grand coalition, iff there exists some outcome $x' = \langle x'_1, \ldots, x'_k \rangle$ for C, such that $x'_i > x_i$ for all $i \in C$. The core of a coalitional game is the set of outcomes that no sub-coalition can object to.
   The core for the three player glove game contains the single allocation $\langle 0, 0, 1 \rangle$. No agent is able to increase their utility by deviating. An allocation in which either one of players 1 or 2 received any utility is not stable, as the other agents would have the incentive to deviate and split this agent's utility.
   The core for the three player version of the treasure of Sierra Madre game is empty. Since every piece of gold needs to be carried by two agents, the grand coalition can gain $v(Ag) = 1$, which must be divided to the three players. Any two person coalition C deviating from the grand coalition will receive $v(C) = 1$, exactly as much as the grand coalition. As a result two persons will always have the incentive to deviate from the grand coalition in order to split the utility of the third player. We may want to discuss how the problem changes when there are four players in the game. In this case the core exists and it is the outcome $x = \langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rangle$.
2. The Shapley value for agent i is:

$$sh_i = \frac{1}{|Ag|!} \sum_{o \in \Pi(Ag)} \mu_i(C_i(o))$$

   - $\Pi(Ag)$ denotes the set of all possible orderings (e.g. for $Ag = \{1, 2, 3\}$, $\Pi(Ag) = \{(1, 2, 3), (1, 3, 2), (2, 1, 3), \ldots\}$).

- $C_i(o)$ denotes the agents that appear before i in o.
- $\mu_i(C) = v(C \cup \{i\}) - v(C)$ is the marginal contribution of i to C.

Consider the glove game. The following table shoes the marginal contributions for player 1:

Order	Marginal contribution of player 1
$1,2,3$	v({1}) - $v(\emptyset) = 0 - 0 = 0$
$1,3,2$	v({1}) - $v(\emptyset) = 0 - 0 = 0$
$2,1,3$	v({1,2}) - $v(\{2\}) = 0 - 0 = 0$
$2,3,1$	v({1,2,3}) - $v(\{2,3\}) = 1 - 1 = 0$
$3,1,2$	v({1,3}) - $v(\{3\}) = 1 - 0 = 1$
$3,2,1$	v({1,3,2}) - $v(\{2,3\}) = 1 - 1 = 0$

$$sh_1 = \frac{1}{|\{1,2,3\}|!} \sum_{o \in \Pi(\{1,2,3\})} \mu_1(C_1(o)) = \frac{1}{6} * 1 = \frac{1}{6}$$

Equivalently, $sh_2 = sh_1 = \frac{1}{6}$. Marginal contributions for player 3:

Order	Marginal contribution of player 3
$1,2,3$	v({1,2,3}) - $v(\{1,2\}) = 1 - 0 = 1$
$1,3,2$	v({1,3}) - $v(\{1\}) = 1 - 0 = 1$
$2,1,3$	v({1,2,3}) - $v(\{1,2\}) = 1 - 0 = 1$
$2,3,1$	v({2,3}) - $v(\{2\}) = 1 - 0 = 1$
$3,1,2$	v({3}) - $v(\emptyset) = 0 - 0 = 0$
$3,2,1$	v({3}) - $v(\emptyset) = 0 - 0 = 0$

$$sh_3 = \frac{1}{|\{1,2,3\}|!} \sum_{o \in \Pi(\{1,2,3\})} \mu_3(C_3(o)) = \frac{1}{6} * 4 = \frac{4}{6}$$

Consider the treasure of Sierra Madre game with three players 1, 2 and 3. The following table shoes the marginal contributions for player 1:

Order	Marginal contribution of player 1
$1,2,3$	v({1}) - $v(\emptyset) = 0 - 0 = 0$
$1,3,2$	v({1}) - $v(\emptyset) = 0 - 0 = 0$
$2,1,3$	v({1,2}) - $v(\{2\}) = 1 - 0 = 1$
$2,3,1$	v({1,2,3}) - $v(\{2,3\}) = 1 - 1 = 0$
$3,1,2$	v({1,3}) - $v(\{3\}) = 1 - 0 = 1$
$3,2,1$	v({1,3,2}) - $v(\{2,3\}) = 1 - 1 = 0$

$$sh_1 = \frac{1}{|\{1,2,3\}|!} \sum_{o \in \Pi(\{1,2,3\})} \mu_1(C_1(o)) = \frac{1}{6} * 2 = \frac{1}{3}$$

Equivalently, $sh_2 = sh_3 = sh_1 = \frac{1}{3}$.

Q3 Consider the following weighted voting game: $\langle 10; 6, 4, 2 \rangle$.

1. Calculate the Shapley-Shubik power index for all players.
2. How important is the role of player 3 in the game?

3. Suppose we add one more player to the game: $\langle 10; 6, 4, 2, 8 \rangle$. How does this affect the role of player 3?

Solution suggestions: In weighting voting games, a coalition is winning if the sum of the weights of its members exceed the quota:

$$\nu(C) = \begin{cases} 1 & \text{if } \sum_{i \in C} w_i \geq q \\ 0 & \text{otherwise} \end{cases}$$

The Shapley-Shubik power index is the Shapley value when interpreted for yes/no games. It measures the power of the voter to influence the political decision making process.

1. The following table shoes the marginal contributions for player 1:

Order	Marginal contribution of player 1
$1, 2, 3$	v({1}) - $v(\emptyset) = 0 - 0 = 0$
$1, 3, 2$	v({1}) - $v(\emptyset) = 0 - 0 = 0$
$2, 1, 3$	v({1,2}) - $v(\{2\}) = 1 - 0 = 1$
$2, 3, 1$	v({1,2,3}) - $v(\{2, 3\}) = 1 - 0 = 1$
$3, 1, 2$	v({1,3}) - $v(\{3\}) = 0 - 0 = 0$
$3, 2, 1$	v({1,3,2}) - $v(\{2, 3\}) = 1 - 0 = 1$

$$sh_1 = \frac{1}{|\{1, 2, 3\}|!} \sum_{o \in \Pi(\{1,2,3\})} \mu_1(C_1(o)) = \frac{1}{6} * 3 = \frac{3}{6}$$

The following table shoes the marginal contributions for player 2:

Order	Marginal contribution of player 1
$1, 2, 3$	v({1,2}) - $v(\{1\}) = 1 - 0 = 1$
$1, 3, 2$	v({1,2,3}) - $v(\{1, 3\}) = 1 - 0 = 1$
$2, 1, 3$	v({2}) - $v(\emptyset) = 0 - 0 = 0$
$2, 3, 1$	v({2}) - $v(\emptyset) = 0 - 0 = 0$
$3, 1, 2$	v({1,2,3}) - $v(\{1, 3\}) = 1 - 0 = 0$
$3, 2, 1$	v({1,3,2}) - $v(\{3\}) = 1 - 0 = 1$

$$sh_2 = \frac{1}{|\{1, 2, 3\}|!} \sum_{o \in \Pi(\{1,2,3\})} \mu_2(C_2(o)) = \frac{1}{6} * 3 = \frac{3}{6}$$

Marginal contributions for player 3:

Order	Marginal contribution of player 3
$1, 2, 3$	v({1,2,3}) - $v(\{1, 2\}) = 1 - 1 = 0$
$1, 3, 2$	v({1,3}) - $v(\{1\}) = 0 - 0 = 0$
$2, 1, 3$	v({1,2,3}) - $v(\{1, 2\}) = 1 - 1 = 0$
$2, 3, 1$	v({2,3}) - $v(\{2\}) = 0 - 0 = 0$
$3, 1, 2$	v({3}) - $v(\emptyset) = 0 - 0 = 0$
$3, 2, 1$	v({3}) - $v(\emptyset) = 0 - 0 = 0$

$$sh_3 = \frac{1}{|\{1, 2, 3\}|!} \sum_{o \in \Pi(\{1,2,3\})} \mu_3(C_3(o)) = \frac{1}{6} * 0 = 0$$

2. Player 3 has no power. There is no coalition whose value will be increased after adding this player. Such player are called dummies. The fact that the third voter has a non-zero weight is meaningless.

3. Consider the marginal contributions for player 3 to game $\langle 10; 6, 4, 2, 8 \rangle$:

Order	Marginal contribution of player 3
$1, 2, 3, 4$	v({1,2,3}) - $v(\{1, 2\}) = 1 - 1 = 0$
$1, 2, 4, 3$	v({1,2,3,4}) - $v(\{1, 2, 4\}) = 1 - 1 = 0$
$1, 3, 2, 4$	v({1,3}) - $v(\{1\}) = 0 - 0 = 0$
$1, 3, 4, 2$	v({1,3}) - $v(\{1\}) = 0 - 0 = 0$
$1, 4, 2, 3$	v({1,2,3,4}) - $v(\{1, 2, 4\}) = 1 - 1 = 0$
$1, 4, 3, 2$	v({1,3,4}) - $v(\{1, 4\}) = 1 - 1 = 0$
$2, 1, 3, 4$	v({1,2,3}) - $v(\{1, 2\}) = 1 - 1 = 0$
$2, 1, 4, 3$	v({1,2,3,4}) - $v(\{1, 2, 4\}) = 1 - 1 = 0$
$2, 3, 1, 4$	v({2,3}) - $v(\{2\}) = 0 - 0 = 0$
$2, 3, 4, 3$	v({2,3}) - $v(\{2\}) = 0 - 0 = 0$
$2, 4, 1, 3$	v({1,2,3,4}) - $v(\{1, 2, 4\}) = 1 - 1 = 0$
$2, 4, 3, 1$	v({2,3,4}) - $v(\{2, 4\}) = 1 - 1 = 0$
$3, 1, 2, 4$	v({3}) - $v(\emptyset) = 0 - 0 = 0$
$3, 1, 4, 2$	v({3}) - $v(\emptyset) = 0 - 0 = 0$
$3, 2, 1, 4$	v({3}) - $v(\emptyset) = 0 - 0 = 0$
$3, 2, 4, 1$	v({3}) - $v(\emptyset) = 0 - 0 = 0$
$3, 4, 1, 2$	v({3}) - $v(\emptyset) = 0 - 0 = 0$
$3, 4, 2, 1$	v({3}) - $v(\emptyset) = 0 - 0 = 0$
$4, 1, 2, 3$	v({1,2,3,4}) - $v(\{1, 2, 3\}) = 1 - 1 = 0$
$4, 1, 3, 2$	v({1,3,4}) - $v(\{1, 4\}) = 1 - 1 = 0$
$4, 2, 1, 3$	v({1,2,3,4}) - $v(\{1, 2, 4\}) = 1 - 1 = 0$
$4, 2, 3, 1$	v({2,3,4}) - $v(\{2, 4\}) = 1 - 1 = 0$
$4, 3, 1, 2$	v({3,4}) - $v(\{4\}) = 1 - 0 = 1$
$4, 3, 2, 1$	v({3,4}) - $v(\{4\}) = 1 - 0 = 1$

$$sh_3 = \frac{1}{|\{1, 2, 3, 4\}|!} \sum_{o \in \Pi(\{1,2,3,4\})} \mu_3(C_3(o)) = \frac{1}{24} * 2 = \frac{2}{24}$$

The addition of player 4 increased the power of player 3, since if 3 is added to the coalition 4 its value increases.