Agent-Based Systems

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Lecture 10 – Coalition Formation

Where are we?

- Discussed procedures for making group decisions
- Simple mechanisms: plurality, sequential majority
- Advanced mechanisms: Borda Count, Slater Ranking
- Desirable properties, paradoxes and dictatorships
- Strategic manipulation and computational complexity

Today ...

- **Forming Coalitions**

Forming Coalitions

- In games like the Prisoner's Dilemma cooperation is prevented because:
  - Binding agreements are not possible
  - Utility is given directly to individuals as the result of individual action
- These features do not hold in many real world situations:
  - Contracts can form binding arrangements
  - Revenue that a company earns is not credited to an individual
- When we lift these assumptions cooperation is both possible and rational
- **Cooperative game theory** asks which contracts are meaningful solutions among self-interested agents

Terminology

- $Ag = \{1, \dots, n\}$ agents (typically $n > 2$)
- Any subset C of Ag is called a **coalition**
- $C = Ag$ is the **grand coalition**,
- A **cooperative game** is a pair $\mathcal{G} = \langle Ag, \nu \rangle$
- $\nu : \mathbf{2}^{Ag} \rightarrow \mathbb{R}$ is the **characteristic function** of the game
- $\nu(C)$ is the utility C can achieve, regardless of $Ag - C$'s behaviour
- **Singleton coalitions** contain one agent (describe what agents can achieve alone)
- Neither individual actions and utilities matter, nor the origin of ν

Three Stages of Cooperative Action

- Coalition structure generation
  - Asking which coalitions will form, concerned with **stability**
  - For example, a productive agent has the incentive to defect from a coalition with a lazy agent
  - Necessary but not sufficient condition for establishment of a coalition
- Solving the optimisation problem of each coalition
  - Decide on collective plans
  - Maximise the collective utility of the coalition
- Dividing the value of the solution of each coalition
  - Concerned with **fairness** of contract
  - How much an agent should receive based on her contribution

Outcomes and Objections

- An **outcome** $x = \langle x_1, \ldots, x_k \rangle$ for a coalition C in game $\langle Ag, \nu \rangle$ is a distribution of C's utility to members of C
- Outcomes must be **feasible** (don't overspend) and **efficient** (don't underspend): $\sum_{i \in C} x_i = \nu(C)$
- Example:
  - $Ag = \{1, 2\}$, $\nu(\{1\}) = 5$, $\nu(\{2\}) = 5$ and $\nu(\{1, 2\}) = 20$
  - Possible outcomes for $C = \{1, 2\}$ are $\langle 20, 0 \rangle$, $\langle 19, 1 \rangle$, $\ldots$, $\langle 0, 20 \rangle$
- *C* **objects** to an outcome for the grand coalition if there is some outcome for C in which all members of C are strictly better off
- Formally, $C \subseteq Ag$ objects to $x = \langle x_1, \ldots, x_n \rangle$ for the grand coalition, iff there exists some outcome $x' = \langle x'_1, \ldots, x'_k \rangle$ for C, such that $x'_i > x_i$ for all $i \in C$

The Core

- The **core** of a coalitional game is the set of outcomes that no sub-coalition can object to
- If the core is non-empty, then the grand coalition is stable
- The core of the previous example contains all outcomes between $\langle 15, 5 \rangle$ and $\langle 5, 15 \rangle$ inclusive
- Problems:
  - Sometimes the core is empty
  - **Fairness:** $\langle 15, 5 \rangle$ distributes all the surplus generated by the cooperation to one agent (fairness?)
  - The definition of the core involves quantification over all possible coalitions, so all of them have to be enumerated

The Shapley Value (I)

- To eliminate unfair distribution, try to divide surplus according to contribution
- Define marginal contribution of i to C: $\mu_i(C) = \nu(C \cup \{i\}) - \nu(C)$
- Axioms any fair distribution should satisfy:
  - **Symmetry:** if two agents contribute the same they should receive the same pay-off (they are interchangeable)
  - **Dummy player:** agents that do not add value to any coalition should get what they earn on their own
  - **Additivity:** if two games are combined, the value a player gets should be the sum of the values it gets in individual games

The Shapley Value (I)

- The **Shapley value** for agent i :
$$sh_i = \frac{1}{|Ag|!} \sum_{o \in \Pi(Ag)} \mu_i(C_i(o))$$
  - $\Pi(Ag)$ denotes the set of all possible orderings (e.g. for $Ag = \{1,2,3\}$, $\Pi(Ag) = \{(1,2,3),(1,3,2),(2,1,3),\ldots\}$)
  - $C_i(o)$ denotes the agents that appear before i in o
- Requires that
  - $\nu(\emptyset) = 0$ and
  - $\nu(C \cup C') \geq \nu(C) + \nu(C')$ if $C \cap C' = \emptyset$

  (ν **superadditive**)
- Strong result: The Shapley value is the *only* value that satisfies the fairness axioms

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Representation

- A naive representation of a coalition game is infeasible (exponential in the size of Ag):

```
1, 2, 3
1 = 5
2 = 5
3 = 5
1, 2 = 10
1, 3 = 10
2, 3 = 20
1, 2, 3 = 25
```

- As with preference orderings, we need a **succinct** representations
- **Modular representations** exploit Shapley's axioms directly
- Basic idea: divide the game into smaller games and exploit additivity axiom

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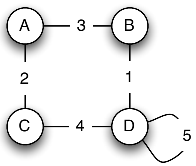
Induced Subgraphs

- Define a characteristic function by an undirected weighted graph
- Value of a a coalition $C \subseteq Ag$: $\nu(C) = \sum_{\{i,j\} \subseteq C} w_{i,j}$
- Example:



$\nu(\{A,B,C\}) = 3 + 2 = 5$
$\nu(\{D\}) = 5$
$\nu(\{B,D\}) = 1 + 5 = 6$
$\nu(\{A,C\}) = 2$

- Not a complete representation (not all characteristic functions can be represented)
- But easy to compute the Shapley value for a given player in polynomial time
  - $sh_i = \frac{1}{2} \sum_j w_{i,j}$
- Checking emptiness of the core is NP-complete, and membership to the core is co-NP-complete

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Marginal Contribution Nets

- Represent characteristic function as rules: pattern $\longrightarrow$ value
  - the pattern is a conjunction of agents, e.g. $1 \wedge 3$
  - $1 \wedge 3$ would apply to $\{1,3\}$ and $\{1,3,5\}$, but not to $\{1\}$ or $\{8,12\}$
  - $C \models \varphi$, means the rule $\varphi \longrightarrow x$ applies to coalition C
  - $rs_C = \{\varphi \longrightarrow x \in rs | C \models \varphi\}$ are the rules that apply to coalition C
- $\nu_{rs}(C) = \sum_{\varphi \longrightarrow x \in rs_C} x$
- Example:
  - $rs_1 = \{a \wedge b \longrightarrow 5,\ b \longrightarrow 2\}$
  - $\nu_{rs_1}(\{a\}) = 0$, $\nu_{rs_1}(\{b\}) = 2$ and $\nu_{rs_1}(\{a,b\}) = 7$
- Extension: allow negation in rules, e.g. $b \wedge \neg c \longrightarrow -2$
- Shapley value can be computed in polynomial time
- Complete representation, but not necessarily succinct

Representations for Simple Games

- A coalitional game is **simple** if the value of any coalition is either 0 (losing) or 1 (winning)
- Simple games model **yes/no** voting systems
- $Y = \langle Ag, W \rangle$, where $W \subseteq \mathbf{2}^{Ag}$ is the set of winning coalitions
- If $C \in W$, C would be able to determine the outcome, 'yes' or 'no'
- Important conditions:
  - **Non-triviality:** $\emptyset \subset W \subset \mathbf{2}^{Ag}$
  - **Monotonicity:** if $C_1 \subseteq C_2$ and $C_1 \in W$ then $C_2 \in W$
  - **Zero-sum:** if $C \in W$ then $Ag \backslash C \notin W$
  - **Empty coalition loses:** $\emptyset \notin W$
  - **Grand coalition wins:** $Ag \in W$
- Naive representation is exponential in the number of agents

Weighted Voting Games

- For each agent $i \in Ag$ define a weight w_i and an overall **quota** q
- A coalition is winning if the sum of their weights exceeds the quota:
  $$\nu(C) = \begin{cases} 1 & \text{if } \sum_{i \in C} w_i \geq q \\ 0 & \text{otherwise} \end{cases}$$
- Example: **Simple majority voting**, $w_i = 1$ and $q = \frac{\lceil |Ag|+1 \rceil}{2}$
- Succinct (but incomplete) representation: $\langle q; w_1, \ldots, w_n \rangle$
- Extension: ***k*-weighted voting games** are a complete representation
  - overall game = "conjunction" k of k different weighted voting games
  - Winning coalition is the one that wins in all component games
  - **Game dimension**: k is at most exponential in the number of players
  - Checking whether a k-weighted voting game is minimal is NP-complete

Weighted Voting Games (II)

- **Shapley-Shubic power index** = Shapley value in yes/no games
  - Measures the power of the voter in this case
  - Computation is NP-hard, no reasonable polynomial time approximation
  - Checking emptiness of the core can be done in polynomial time (*veto player)*
- Counter-intuitive properties:
  - In $\langle 100; 99, 99, 1 \rangle$, all voters have the same power $(\frac{1}{3})$
  - Dummy with non-zero power, e.g. $\langle 10; 6, 4, 2 \rangle$, meaningful?
  - Adding new voters increases voter power, e.g. $\langle 10; 6, 4, 2, 8 \rangle$

Summary

- Coalition formation
- The core and the Shapley value
- Different representations
- Simple games
- Next time: **Resource Allocation**