ABS 2011-12 stuff

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1 Set-theoretic notation, functions and relations

- A set $A = \{a_1, \dots, a_r\}$ is a collection of elements $a_1, \dots, a_r$, it may be finite or infinite. Sets contain each element only once. The empty set is written as $\emptyset$. Examples: $\mathit{Fruits} = \{\mathit{Apple}, \mathit{Orange}\}$, the set of natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$.
- We write $a \in S$ if a is contained in the set S, $a \notin S$ else. The cardinality (size) of a set S is denoted by $|S|$, e.g. $|\{1, 2, 3, 4\}| = 4$, $\emptyset = 0$, $\mathbb{N} = \infty$.
- $S = T$ implies that S and T are equal, i.e. every element of S is in T and every element of T is in S.
- $S \subseteq T$ denotes that the set S is a subset of the set T, i.e. every element of S is also an element of T .
- The powerset $\wp(S)$ (or 2^S) is the set of all subsets of S, e.g. $\wp(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. For any finite set S, $|\wp(S)| = 2^{|S|}$.
- If $\exists a \in S\ .P(a)$, there exists an a in S for which the property P holds. E.g. $\exists a \in \mathbb{N}\ .a \leq 3$, this is true for $a \in \{1, 2, 3\}$.
- If $\forall a \in S\ .P(a)$ property P holds for every element in S . E.g. $\forall a \in \mathbb{N}\ .a \geq 1$, this is true for any $a \in \mathbb{N}$.
- $\{a \in S|P(a)\}$ is the set of all a in S for which P holds (S is sometimes omitted if it can be determined from context). For example, $\{a \in \mathbb{N}|\exists k \in \mathbb{N}\ .a = 2k\}$ is the set of all even natural numbers.
- $A \cup B$ is the union of sets, $A \cup B = \{x|x \in A \text{ or } x \in B\}$, e.g. $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$.
- $A \cap B$ is intersection of sets, $A \cap B = x|x \in A \text{ and } x \in B$, e.g. $\{1, 2\} \cap \{2, 3\} = \{2\}$.
- $A - B$ (or $A \backslash B$) is shorthand notation for the set $\{x \in A|x \notin B\}$.
- $A \times B$ is the Cartesian product (or cross-product) of two sets, $A \times B = \{(a, b)|a \in A, b \in B\}$, for example $\{1, 2\} \times \{2, 3\} = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$. We write

  $$A^n = \underbrace{A \times \ldots \times A}_{n \text{ times}}$$

  for the n-fold cross-product of a set with itself.
- To obtain arbitrary-length ordered lists of objects over a set (which we normally write as $\langle a_1, \dots, a_n \rangle$, $a_i \in S$), we construct the infinite union $A^* = \cup_{i=0}^{\infty} A^i$ (to exclude the empty sequence, we write $A^+ = A^* \backslash \langle\rangle$).
- A relation is a subset of a cross-product over sets, e.g. $R \subseteq A \times B$ for a binary relation, or $R \subseteq A^n$ for an n-ary (reflexive) relation.
- A function is a relation $R \subseteq A \times B$ where every element of A is only associated with *one* element of B, i.e. $\forall a \in A\ .|\{b \in B|(a, b) \in R\}| = 1$.
- If R is a function, we often write $R : A \to B$, and $R(a) = b$ or $a \mapsto b$ to specify which element a is mapped to in B.