Categorical Perspectives

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Abstract. Quantales provide an abstract algebra of actions equipped with a binary operation of sequential composition and an infinitary operation (sup) of non-deterministic amalgamation. Formally, quantales are monoids in the category of complete sup-lattices. Quantales have provided a setting for studying ontic actions and various process equivalences. More recently, they have been used as a semantic setting for discussion of epistemic actions and quantum logics.

The archetypical example is given by the monoid of binary relations on a set S. We think of these as non-deterministic actions, acting on states that are elements of S. It is known that every quantale $\mathfrak{Q}$ may be represented as a quantale of relations—indeed, $\mathfrak{Q}$ has several representations, derived from the Cayley representation of the underlying monoid, as a set of relations on $\mathfrak{Q}$. However, these representations use a subset of relations that is not, in general, closed under suprema, so non-determinism is not faithfully represented.

We seek to interpret $\mathfrak{Q}$ as a quantale of relations over a non-classical set. Given a quantale, $\mathfrak{Q}$, we construct the classifying topos for a set equipped with relations reflecting the structure of $\mathfrak{Q}$. We represent $\mathfrak{Q}$ as a quantale of global sections of relations on the generic object in this classifying topos. This provides a universal, or generic, relational representation of $\mathfrak{Q}$, in the normal sense of classifying topoi.

The site supporting this classifying topos has as objects finitely presented transition systems that represent lax quotients of $\mathfrak{Q}$. We interpret these as perspectives, representing a local focus on some aspects of the world—a finitely-observable set of observations of the effects of some actions, compatible with the structure of $\mathfrak{Q}$. This category is equipped with a Grothendieck topology that forces the representation to be strict.

1 Introduction

A standard model of actions first represents states of a system, then represents actions as binary relations on this set of states—ontology precedes action. From a cognitive perspective, this is puzzling: our mental models of the world must surely be derived from experience of action. Similarly, from a physical perspective, ontology muct be grounded in experiment—a form of action. This poses a problem: How can ontology be infered from actions?

If we abstract away from a simple model of actions, as operations that change state, we get a simple algebraic structure. Mathematically, we can ask for a representation theorem—to what extent does the abstraction capture the structure.

Philosophically, we can take a different view, and ask, "Where does our platonic model of reality come from?" We perform actions. How can we infer platonic models from the structure of possible actions? What different views of the world can there be and how can we reconcile different views to get a coherent picture of reality?

In this note we discuss a simple abstract version of this problem. We review the standard model of state and action underlying many treatments in computer science and artificial intellignece, and describe a construction that generates, or recovers, an ontology from an abstract algebra of actions.

We begin with a brief discussion of actions and quantales: in particular quantales of relations, in the category of sets, $\mathbb{S}$.

Quantales of relations may be defined in elementary topos, and we observe that our $\mathbb{S}$-based discussion is constructive, in the sense that it can be interpreted within an elementary topos.

The global sections of a quantale form a quantale, and some properties of quantales are preserved by the global sections functor.

We present some examples, then describe the construction of a generic representation of a quantale in the quantale of global sections of relations on some sheaf.

2 States and Actions

Let I be a set of *state-variables*, or *fluents* with each $x \in I$ ranging over a non-empty set $\mathcal{S}_x$ of possible values. States are *valuations*—functions that assign a value to each state-variable. So, valuations, $\mathbf{v}$, are elements of the product *state-space.* A *property*, $\varphi \subseteq \mathcal{S}$, is a subset of the state space.

$$\mathbf{v} \in \mathcal{S} = \prod_{x \in I} \mathcal{S}_x \qquad \mathbf{v} = \langle \mathbf{v}_x \in \mathcal{S}_x \mid x \in I \rangle \qquad \varphi \in \wp(\mathcal{S})$$

A non-deterministic *action* is represented by a $\bigcup$-preserving function on properties, $\alpha : \wp(S) \longrightarrow \wp(S)$. This representation has a logical reading: the result[1], $\varphi \cdot \alpha$, is the strongest postcondition consisting of all states that *may be reached, from some state satisfying* φ*, by performing* α.

Equivalently, we may represent α by the relation $\mathbf{w} \in \{\mathbf{v}\} \cdot \alpha$ from states to states, which we write, $\mathbf{v} \xrightarrow{\alpha} \mathbf{w}$, and which has an operational reading: *"doing* α *in state* $\mathbf{v}$ *may lead to state* $\mathbf{w}$*"*.

$$\alpha : \wp(S) \longrightarrow \wp(S) \qquad \varphi \cdot \alpha = \{\mathbf{w} \mid \exists \mathbf{v} \in \varphi .\, \mathbf{v} \xrightarrow{\alpha} \mathbf{w}\}$$

Composition of actions, *qua* functions or relations, represents *sequential composition* (;); *non-deterministic composition* of actions ($\bigvee$) is given by unions:

$$\varphi \cdot (\alpha \,;\, \beta) = (\varphi \cdot \alpha) \cdot \beta \qquad \varphi \cdot \bigvee_i \alpha_i = \bigcup_i (\varphi \cdot \alpha_i)$$

[1] We use a notation of application on the right for these functions.

3 Quantales

Quantales are abstract algebras of actions, equipped with a specialisation ordering ($\alpha \leq \beta$ if β may do everything that α may do) and operations of sequential (;) and non-deterministic ($\bigvee$) composition. We call the elements *actions*. Non-deterministic composition gives joins (least upper bounds) for the specialisation order. For sequential composition, we use multiplicative notation, writing 1 for the identity (the null action which accepts every state and does nothing), and $x\,;y$, or often simply xy, for the composite action, x *then* y.

Definition 1 ([Mul86][2]). *A* quantale*, $\mathfrak{Q}$ is a monoid in the category of $\bigvee$-lattices: that is, a complete sup-lattice equipped with an associative, binary operation (;), that preserves joins ($\bigvee$) in each argument, and has an* identity*, 1, such that $1\,;x = x = x\,;1$:*

$$(x\,;y)\,;z = x\,;(y\,;z) \qquad \Big(\bigvee_i x_i\Big)\,;y = \bigvee_i (x_i\,;y) \qquad x\,;\bigvee_i y_j = \bigvee_j (x\,;y_j)$$

An involution *on $\mathfrak{Q}$ is a map $\mathfrak{Q} \xrightarrow{(-)^\circ} \mathfrak{Q}$ such that:*

$$x^{\circ\circ} = x \qquad (xy)^\circ = y^\circ x^\circ \qquad (\bigvee x_i)^\circ = \bigvee x_i^\circ$$

A quantale equipped with an involution is said to be involutive*. A quantale whose underlying sup-lattice is a frame—which means simply that it is completely distributive: $x \wedge \bigvee y_i = \bigvee (x \wedge y_i)$—is called a* quantal frame*.*

$\mathfrak{Q}$ is regular if it is regular as a semigroup—for each element a, there exists an element x such that $axa = a$.

Example 1. The binary relations, $R \subseteq \mathcal{X} \times \mathcal{X}$, on a set $\mathcal{X}$, ordered by set inclusion and composed by relational composition, form an involutive quantal frame, $\mathfrak{R}(\mathcal{X})$, where $\mathbf{r}^\circ$ is the *reciprocal* of $\mathbf{r}$, defined by $x\,\mathbf{r}^\circ\,y \iff y\,\mathbf{r}\,x$.

Example 2. The (sup-preserving) automorphisms of any sup-lattice Λ, equipped with the pointwise ordering and function composition, form a quantale, $\mathfrak{A}(\Lambda)$.

There is an obvious isomorphism, $\mathfrak{R}(\mathcal{X}) \equiv \mathfrak{A}(\wp(\mathcal{X}))$, between the automorphisms of the powerset and the quantale of relations—take the image of a set under a relation, or apply an automorphism to a singleton set.

The relations on a set, $\mathcal{X}$, form an involutive quantal frame, $\mathfrak{R}(\mathcal{X})$, whose specialisation order is just inclusion. They are generated by subunits ($y = \bigvee\{x \leq y \mid xx^\circ \leq 1\}$), and may be presented as the lattice of sup-closed ideals of the poset of subunits.

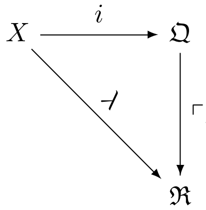
[2] Mulvey (*op. cit.*), and others, use e for the identity, and 1 for the top element of the lattice—which we denote by 1, and $\top$, respectively. All our quantales are *unital* in the sense of Mulvey. However, our key construction (see §6) can also be applied to a general quantale. Note that composition (;) is monotone in each argument, since it preserves $\vee$.

A quantale of relations, $\mathfrak{R}(\mathcal{X})$, has some particular properties. Algebraically, it has an involution, given by taking the transpose of each relation: $x\,\mathbf{r}\,y \vdash y\,\mathbf{r}^\circ\,x$. Furthermore, $\mathbf{r} \le \mathbf{r}\mathbf{r}^\circ\mathbf{r}$, so each $\mathbf{r} \le 1$ is regular. $\wp(\mathcal{X} \times \mathcal{X})$, like any powerset, is an atomic distributive sup-lattice, or atomic frame. Algebraically, the atoms, which are the singletons $\{\langle x, y\rangle\}$, are sub-units. The subunits thus generate the quantale ($y = \bigvee\{x \le y \mid xx^\circ \le 1\}$), which may be presented as the lattice of sup-closed ideals of the poset of subunits. So $\mathfrak{R}(\mathcal{X})$ is an inverse quantale frame, in the sense of Resende [Res07].

Definition 2. *A* (strict) morphism *of quantales* $\mathfrak{Q} \xrightarrow{f} \mathfrak{R}$ *is a map that is both a morphism of sup-lattices, and a monoid homomorphism: it preserves* $\bigvee, ;, 0, 1$. *A* lax morphism *of quantales is a morphism f of sup-lattices such that* $1_{\mathfrak{R}} \le f(1_{\mathfrak{Q}})$ *(f is* 1-lax*) and* $f(\alpha) ; f(\beta) \le f(\alpha ; \beta)$ *(f is* ;-lax*). An* op-lax morphism *of quantales is a morphism f of sup-lattices, such that* $f(1_{\mathfrak{Q}}) \le 1_{\mathfrak{R}}$ *(f is* 1-strict*) and* $f(\alpha ; \beta) \le f(\alpha) ; f(\beta)$ *(f is* ;-strict*).*

Maps from a set X to (the underlying set of) a quantale, $\mathfrak{Q}$ form a sup-lattice under the pointwise partial order, so concrete categories of quantales are naturally enriched (hom-sets form posets, and often sup-lattices). Each of these three classes of morphism (lax, op-lax, strict) is closed under sup. So in each case the hom-sets are naturally sup-lattices. Clearly, lax morphisms are also closed under meets $\bigwedge$, so we can make the following definition.

Definition 3. *Given quantales,* $\mathfrak{Q}$, $\mathfrak{R}$, *a set* $\Lambda \xrightarrow{i} \mathfrak{Q}$ *of actions in* $\mathfrak{Q}$, *and a map* $\Lambda \xrightarrow{\lambda} \mathfrak{R}$, *we define* $\mathfrak{Q} \xrightarrow{\ulcorner\lambda\urcorner} \mathfrak{R}$, *the* extension of λ along i, *to be the minimal lax morphism such that* $\lambda \le i \circ \ulcorner\lambda\urcorner$.



$$\ulcorner\lambda\urcorner = \bigwedge\{\rho \mid \mathfrak{Q} \xrightarrow[lax]{\rho} \mathfrak{R} \text{ and } \lambda \le \rho\}$$

We will use this construction later.

Example 3. Given a function $X \xrightarrow{f} Y$, we have three order-preserving maps

$$\wp(X) \;\begin{array}{c} \xrightarrow{\forall_f} \\ \xleftarrow{f^*} \\ \xrightarrow{\exists_f} \end{array}\; \wp(Y)$$

These form a stack of adjoints, $\exists_f \dashv f^* \dashv \forall_f$. As left adjoints preserve limits (in this setting, suprema), the two left adjoints form an adjoint pair of sup-lattice morphisms.

Given $\mathcal{X} \xrightarrow{f} \mathcal{Y}$, the image, $\wp(\mathcal{X} \times \mathcal{X}) \xrightarrow{\exists_{f\times f}} \wp(\mathcal{Y} \times \mathcal{Y})$, acts as an op-lax quantale morphism $\mathfrak{R}(\mathcal{X}) \xrightarrow{f} \mathfrak{R}(\mathcal{Y})$; the inverse image $\wp(\mathcal{Y}\times\mathcal{Y}) \xrightarrow{(f\times f)^*} \wp(\mathcal{X} \times \mathcal{X})$ is a lax quantale morphism.

4 Perspectives

We introduce the idea of a family $U \in \mathcal{O}$ of *perspectives*—these may be thought of as experiments or observations. The intuition is that these "perspectives" should represent a local focus on some aspects of the world—an isolated environment within which we may experiment independently of outside influences. We write $\mathcal{P}(U)$ for the local states that may be hypothesized and, potentially, observed from this perspective U. One observation, U, may be included in, or derived from, another, V. We represent this change of perspective by an arrow $U \longrightarrow V$. A corresponding map, $\mathcal{P}(V) \longrightarrow \mathcal{P}(U)$, in the other direction, represents the re-presentation of results of V from the perspective of U.

The reader familiar with topoi will recognise this as a presheaf on a category of perspectives. In this context, it is also natural to think that a category of perspectives should come equipped with a notion of covering—a family $U_i \longrightarrow V$ of perspectives covers V if every coherent family of outcomes, $p_i \in \mathcal{P}(U_i)$ is compatible with a unique common extension $\bar{p} \in \mathcal{P}(V)$. This leads us to consider sheaves.

5 Topos Models

A *topos* is an abstract category of sets. The construction and properties of quantales of relations discussed above can be formalised entirely constructively—in the sense that it can be interpreted in any topos. If $\mathbb{E} \xrightarrow{\Gamma} \mathbb{B}$ is a topos over a "base" topos $\mathbb{B}$, and $\mathfrak{Q}$ is a quantale in $\mathbb{E}$ then $\Gamma\mathfrak{Q}$ is a quantale in $\mathbb{B}$. For any object $\mathcal{X}$ of $\mathbb{E}$, the object $\wp(\mathcal{X} \times \mathcal{X})$ of relations on $\mathcal{X}$ is a quantale, $\mathfrak{R}(\mathcal{X})$ in $\mathbb{E}$. So we can construct quantales in $\mathbb{B}$ by taking global sections of quantales of relations in $\mathbb{E}$, or represent quantales in $\mathbb{B}$ as global sections of a quantale of relations in $\mathbb{E}$.

In particular, an object $\mathcal{X}$ of a Grothendieck topos $Sh(C) \longrightarrow \mathbb{S}$ is a sheaf on the site C. The object $\wp(\mathcal{X} \times \mathcal{X})$ with its powerset ordering and relational composition (given by internal existential quantification) is an internal quantale. Just as in sets, it is internally isomorphic to the quantale of sup-preserving functions $\wp(\mathcal{X}) \longrightarrow \wp(\mathcal{X})$. Its global sections correspond to subobjects of $\mathcal{X} \times \mathcal{X}$. These form a quantale in $\mathbb{S}$ (see §6.1).

Example 4 (Presheaves). A pre-sheaf on $\mathbf{C}$ is a functor $A \in \mathcal{S}^{\mathbf{C}^{\mathrm{op}}}$. For $f : Y \longrightarrow X$ in $\mathbf{C}$ we have sets $A(X), A(Y)$ and restriction maps $\upharpoonright_f : A(X) \longrightarrow A(Y)$ that are functorial, $\upharpoonright_{(f \circ g)} = \upharpoonright_f \,;\, \upharpoonright_g$.[3]

If A, B are presheaves on $\mathbf{C}$, we describe the function-space presheaf B^A of functions $A \longrightarrow B$. An element F of the set $B^A(X)$ is a family of functions

$$F_f : A(Y) \longrightarrow B(Y), \text{ indexed by arrows } Y \xrightarrow{f} X \text{ in } \mathbf{C}$$

satisfying the *naturality condition*, that $(F_f y) \upharpoonright_g = F_{f \circ g}(y \upharpoonright_g)$. Restrictions are given by $(F \upharpoonright_f)_g = F_{f \circ g}$.

[3] We write restriction with right-application.

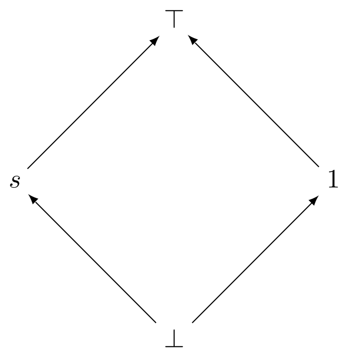
Consider the powerset pre-sheaf, $\wp(A) = \Omega^A$, of a presheaf A. An element U of $\wp(A)(X)$ is given by a family of subsets $U_f \subseteq \wp(A(Y))$, indexed by morphisms $f : Y \longrightarrow X$ in $\mathbf{C}$, such that $x \in U_f \Rightarrow x \upharpoonright_g \in U_{f \circ g}$. The restriction maps are given by $(U \upharpoonright_f)_g = U_{f \circ g}$, and are thus functorial by construction.

Example 5. Let M be a monoid, a category with one object, M; as arrows, we take elements $a, b, \ldots$, of M composing thus: $M \xleftarrow{a} M \xleftarrow{b} M = M \xleftarrow{ab} M$. The presheaves form the category of M-sets: an object is a set, $\mathcal{X} = X(M)$, equipped with a right M-action. The global sections of an object are its M-invariant elements. Products are $\mathbb{S}$-products, with point-wise action: $\langle x, y \rangle g = \langle xg, yg \rangle$. Global relations between M-sets are given by relations on the underlying sets that are invariant under the action. The powerset object is given by families $\mathcal{X}'_f \rightarrowtail \mathcal{X}$ such that

So, for an M-set $\mathcal{X}$, we have the internal quantale $\mathfrak{R}(\mathcal{X}) = \wp(\mathcal{X} \times \mathcal{X})$. The global sections of $\mathfrak{R}(\mathcal{X})$ correspond to subobjects of $\mathcal{X} \times \mathcal{X}$; that is, to subsets of $\mathcal{X} \times \mathcal{X}$ in $\mathbb{S}$ invariant under the action of M.

Example 6. The category, $\mathbb{S}^{\mathbb{Z}_2}$, of sets equipped with an automorphism of order 2, is a topos. Consider the object $\mathcal{Z}_2$, a two-element set, $\{a, b\}$, with the non-trivial $\mathbb{Z}_2$-action.

$\mathcal{Z}_2$ is a two-element sheaf, in that it satisfies the statement,
$\models \exists x, y.\ x \neq y \wedge \forall z.\ z = x \vee z = y$
but, it has no global sections.
The quantale of global relations on $\mathcal{Z}_2$ has four elements, as shown.
The non-trivial action, s, swaps a and b. It satisfies the equations, $s = s^\circ$ and $s^2 = 1$.



This quantale of global sections of the quantale of relations on the generic $\mathbb{Z}_2$-set is precisely the quantale used by Brown and Gurr [BG91] as an example of a non-relational quantale.

An extension of this construction, given by taking global sections of relations on $1 + \mathcal{Z}_2$, provides an example of a quantale not generated by its subunits.

Example 7. We construct $1 + \mathcal{Z}_2$ by adjoining an element, c, (with trivial $\mathbb{Z}_2$ action) to the generic $\mathbb{Z}_2$-set we have two atoms in the lattice of invariant subsets: one $\{a, b\}$ with a non-trivial automorphism; the other $\{c\}$.

Now consider an invariant relation, $\mathbf{r}$. It must satisfy the following constraints:

$$a\,\mathbf{r}\,b \iff b\,\mathbf{r}\,a \quad c\,\mathbf{r}\,a \iff c\,\mathbf{r}\,b \quad a\,\mathbf{r}\,c \iff b\,\mathbf{r}\,c$$

So the quantale of global relations has the two atoms as before, augmented with three new atomic elements $1_c = \{\langle c,c\rangle\}, p = \{\langle a,c\rangle, \langle b,c\rangle\}, p^\circ = \{\langle c,a\rangle, \langle c,b\rangle\}$. The identity is the join of the two partial units $1 = 1_{a,b} \wedge 1_c$. Again, p° is an inverse for p, but neither is a join of partial units. So this is not an inverse quantale in the sense of Resende.

Example 8. The category $Sh(X)$ of sheaves on a topological space, X, is a topos. Consider $Sh(\mathbb{B})$, where $\mathbb{B}$ is the Baire space, of irrational real numbers. Let $\mathcal{X}$ be the sheaf of continuous real-valued functions on $\mathbb{B}$, and U an open of $\mathbb{B}$. So $\mathcal{X}(U)$ is the set of continuous functions $a : U \longrightarrow \mathbb{R}$.

Define actions, α, β, by:

$$\alpha(U) = \left\{ \langle a,b\rangle \;\middle|\; b(x) = \begin{array}{ll} x<0 & \text{if } a(x) \\ x>0 & \text{if } a(x)+x \end{array} \right\}$$

$$\beta(U) = \left\{ \langle a,b\rangle \;\middle|\; b(x) = \begin{array}{ll} x<0 & \text{if } a(x)+x \\ x>0 & \text{if } a(x) \end{array} \right\}$$

then, $(\alpha \vee \beta)(U) = 1 \cup \alpha(U) \cup \beta(U) \cup \{\langle a,b\rangle \mid b(x) = a(x)+x\}$.

These examples show that quantales arising as global sections of relational quantales in a topos are more general than relational quantales in $\mathbb{S}$. They are, nevertheless, involutive quantale frames.

6 Generic Model

We apply a general construction from topos theory. A so-called *geometric theory* has a *classifying topos*, that contains a generic model for the theory. We apply this to produce a generic relational representation of a given quantale. So, starting from a quantale, $\mathfrak{Q}$, we formulate a logical theory whose models correspond to relational representations of $\mathfrak{Q}$.

Let $\mathcal{L}(\mathfrak{Q})$ be the geometric $(\wedge, \bigvee, \exists)$ language with a binary relation symbol $\mathbf{r}$ for each element r of $\mathfrak{Q}$. This means that $\mathcal{L}(\mathfrak{Q})$ is a purely relational predicate logic with: binary relations, $\mathbf{r}$(for each $r \in \mathfrak{Q}$), and $=$; finitary conjunction, $\top, \wedge$; infinitary disjunction, $\bigvee$; and existential quantification, $\exists$.

A model of $\mathcal{L}(\mathfrak{Q})$ in $\mathbb{S}$, the topos of sets, is given by a set $\mathcal{X}$ and a map $[\![-]\!] : \mathfrak{Q} \longrightarrow \wp(\mathcal{X} \times \mathcal{X})$. A model in a Grothendieck topos $\mathbb{G} \xrightarrow{\pi} \mathbb{S}$ is given similarly, by an object $\mathcal{X}$ and a morphism $[\![-]\!] : \pi^*\mathfrak{Q} \longrightarrow \wp(\mathcal{X} \times \mathcal{X})$ in $\mathbb{G}$, or equivalently, a map $\mathfrak{Q} \longrightarrow \mathbb{G}[1, \wp(\mathcal{X} \times \mathcal{X})]$, or a $\mathfrak{Q}$-indexed family of subobjects, $[\![\mathbf{r}]\!] \rightarrowtail \mathcal{X} \times \mathcal{X}$, where $r \in \mathfrak{Q}$.

We recall that $[\![-]\!]$ can be extended, by induction on the structure of φ, to define an *interpretation*, as a subobject $[\![\varphi]\!]_{\bar{x}} \rightarrowtail \mathcal{X}^{\bar{x}}$, for each well-formed formula (wff) φ, and each sequence $\bar{x}$, of distinct variables of $\mathcal{L}$, that contains the free variables of φ.[4]

$$
\begin{array}{rcll}
[\![x = y]\!]_{\langle x,y\rangle} & = & \mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times \mathcal{X} & (1) \\
[\![x\, \mathbf{r}\, y]\!]_{\langle x,y\rangle} & = & [\![r]\!] \longrightarrow \mathcal{X} \times \mathcal{X} & (2) \\
[\![\varphi \wedge \psi]\!]_{\bar{x}} & = & [\![\varphi]\!]_{\bar{x}} \wedge [\![\psi]\!]_{\bar{x}} & (3) \\
[\![\bigvee_i \varphi_i]\!]_{\bar{x}} & = & \bigvee_i [\![\varphi_i]\!]_{\bar{x}} & (4) \\
[\![\exists x.\, \varphi]\!]_{\bar{y}} & = & \exists^x_{\bar{y}} [\![\varphi]\!]_{x,\bar{y}} & (5) \\
[\![\varphi]\!]_{\bar{y}} & = & \pi^*_\sigma [\![\varphi]\!]_{\bar{x}} \qquad \textit{where } \bar{x} \xrightarrow{\sigma} \bar{y} & (6)
\end{array}
$$

We fix an interpretation, and consider *sequents* of the form $\varphi \vdash \psi$. We say the sequent $\varphi \vdash \psi$ is *valid* (written, $\varphi \vDash \psi$) iff $[\![\varphi]\!]_{\bar{x}} \subseteq [\![\psi]\!]_{\bar{x}}$, whenever $\bar{x}$ contains the free variables of φ and ψ.

6.1 Geometric and Cartesian Properties

A *geometric* formula is built from atoms using the connectives $\wedge, \bigvee, \exists$. A *cartesian* formula is built using $\wedge, \exists!$, where $\exists!$ is unique existence. Geometric formulae are preserved by inverse images, cartesian formulae by direct images, or global sections.

Quantales are cartesian, in the sense that they can be axiomatised by entailments between cartesian formulae, so the global sections of a quantale form a quantale in **Set**. Similarly, every quantale of sections of a relational quantale is a regular quantal frame.

Quantales of relations are, internally, inverse quantal frames. However, their global sections can be more general.

Lemma 1. *The interpretation* $[\![-]\!]$ *is*

$$
\begin{array}{rlcr}
\text{order-preserving} & \textit{iff} & x\,\mathbf{r}\,y \vDash x\,\mathbf{s}\,y \qquad \textit{whenever } r \le s & (7) \\
\text{involutive} & \textit{iff} & x\,\mathbf{r}\,y \vDash y\,\mathbf{r}^\circ\,x & (8) \\
\mathbf{;}\text{-lax} & \textit{iff} & x\,\mathbf{r}\,y \wedge y\,\mathbf{s}\,z \vDash x\,(\mathbf{rs})\,z & (9) \\
\text{1-lax} & \textit{iff} & \vDash x\,\mathbf{1}\,x & (10) \\
\text{sup-preserving} & \textit{iff} & x(\bigvee_i \mathbf{s}_i)y \vDash \bigvee_i (x\,\mathbf{s}_i\,y) & (11) \\
\mathbf{;}\text{-strict} & \textit{iff} & x\,(\mathbf{rs})\,z \vDash \exists y.\,(x\,\mathbf{r}\,y \wedge y\,\mathbf{s}\,z) & (12) \\
\text{1-strict} & \textit{iff} & x\,\mathbf{1}\,y \vDash x = y & (13)
\end{array}
$$

The first four axioms, $\mathfrak{A}$ (here 7-10) are *algebraic* in form. The remaining three $\mathfrak{G}$ are *geometric*. The classifying topos, containing a generic relational representaiton of $\mathfrak{Q}$, is constructed by starting with presheaves on $FP(\mathfrak{A})^{\mathrm{op}}$, the opposite

[4] We write $\mathcal{X}^{\bar{y}} \xrightarrow{\pi_\sigma} \mathcal{X}^{\bar{x}}$ for the projection corresponding to a "substitution" map $\bar{x} \xrightarrow{\sigma} \bar{y}$, and $\exists^x_{\bar{y}}$ for $\exists_{\pi_\sigma}$ where σ is the inclusion $\bar{y} \rightarrowtail x, \bar{y}$.

of the category of finitely presented models of $\mathfrak{A}$. This contains a generic lax representation of $\mathfrak{Q}$. The geometric axioms, $\mathfrak{G}$ are then *forced* by introducing a Grothendieck topology.

Finitely-presented models of $\mathfrak{A}$ correspond to finite transition systems, whose transitions are labelled with (possibly many) actions from $\mathfrak{Q}$. So these are our initial candidates for perspectives. Global states arise as global sections—which correspond to mutually consistent sets of hypotheses from each local perspective.

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