Generic Models for Computational Effects

John Power [1]

Laboratory for the Foundations of Computer Science, University of Edinburgh, King's Buildings, Edinburgh EH9 3JZ, Scotland

Abstract

A *Freyd*-category is a subtle generalisation of the notion of a category with finite products. It is suitable for modelling environments in call-by-value programming languages, such as the computational λ-calculus, with computational effects. We develop the theory of *Freyd*-categories with that in mind. We first show that any countable Lawvere theory, hence any signature of operations with countable arity subject to equations, directly generates a *Freyd*-category. We then give canonical, universal embeddings of *Freyd*-categories into closed *Freyd*-categories, characterised by being free cocompletions. The combination of the two constructions sends a signature of operations and equations to the Kleisli category for the monad on the category *Set* generated by it, thus refining the analysis of computational effects given by monads. That in turn allows a more structural analysis of the λ_c-calculus. Our leading examples of signatures arise from side-effects, interactive input/output and exceptions. We extend our analysis to an enriched setting in order to account for recursion and for computational effects and signatures that inherently involve it, such as partiality, nondeterminism and probabilistic nondeterminism.

Key words: *Freyd*-category, enriched Yoneda embedding, conical colimit completion, canonical model

1 Introduction

The notion of *Freyd*-category has emerged over the past ten years as a subtle generalisation of the notion of category with finite products. It allows one to model environments in call-by-value programming languages containing computational effects, notably the λ_c-calculus [25,21,9], a variant of the

Email address: `ajp@inf.ed.ac.uk` (John Power).

[1] This work is supported by EPSRC grant GR/586372/01: A Theory of Effects for Programming Languages.

call-by-value λ-calculus designed specifically to allow one to account for computational effects . Starting with the notion of category with finite products, one obtains the notion of a symmetric monoidal category by dropping insistence upon the existence of diagonals and projections [10]: in such situations, one usually speaks of a tensor product rather than a product, corresponding to the relaxation from cartesian logic to linear logic. If one further drops the insistence upon bifunctoriality of the tensor product, one obtains the notion of a symmetric premonoidal category [25]. This corresponds logically to keeping the terms of linear logic but putting fewer of them equal. Just as one has cartesian closed categories and symmetric monoidal closed categories, one can speak of closedness for a symmetric premonoidal category too [21]. Finally, if one reinstates the assumption of finite product structure but only on a specified subcategory of a putative symmetric premonoidal category, one has the notions of *Freyd*-category and closed *Freyd*-category [9]: we recall the definitions in Section 2. In this paper, motivated by computational effects, we further develop the theory of *Freyd*-categories.

Central to the idea of a computational effect is that of an operation: for global state, one has *lookup* and *update*; for interactive input/output, one has *read* and *write*; for nondeterminism, one has binary $\vee$; etcetera [16,17]. These operations are subject to computationally natural, universally defined equations. Gordon Plotkin and I have studied signatures of such operations extensively (see [18] for a recent summary), and, together with other co-authors, have begun to develop a theory of them (see also [3,4,13,15]). Every signature of operations of countable arity subject to universally defined equations forms a *countable Lawvere theory* L: this is a category with countable products together with structure that forces it to be generated, in a precise sense, by one object, and it is essentially the clone generated by the signature. The first main result of the paper, in Section 3, asserts that the structure of any countable Lawvere theory L yields the structure of a *Freyd*-category on L^{op}. By virtue of its construction, this *Freyd*-category is canonical.

We next seek to embed any small *Freyd*-category into a closed textitFreyd-category, and to do so canonically. For this, we need a variant of the Yoneda embedding [6]. If C is any small category, the Yoneda embedding $Y : C \longrightarrow [C^{op}, Set]$ exhibits $[C^{op}, Set]$ as the free cocompletion of C. If C has finite products, it exhibits $[C^{op}, Set]$ as the free finite product cocompletion of C. And if C is symmetric monoidal, it exhibits $[C^{op}, Set]$ as the free symmetric monoidal cocompletion of C [5]. So we seek to adapt that group of results to the setting of *Freyd*-structure. That requires some work as a *Freyd*-category is not just a single category with structure, but rather involves a pair of categories and an identity-on-objects functor $J : C_0 \longrightarrow C_1$. Thus we need an *enriched* version of the Yoneda embedding [6].

For any cartesian closed (more generally symmetric monoidal closed) category

V, one can define a notion of a category enriched in V, or more briefly, a V-category. In Section 4, we define enrichment and show that if V is the cartesian closed category $[\rightarrow, Set]$, a V-category consists exactly of a pair of categories and an identity-on-objects functor $J : C_0 \longrightarrow C_1$, i.e., the basic data for a *Freyd*-category.

Any cartesian closed category V may itself be seen as V-category, and, under reasonable size and completeness conditions, for every small V-category C, one has a functor V-category $[C^{op}, V]$ and a fully faithful V-functor $Y : C \longrightarrow [C^{op}, V]$, providing a definitive notion of a V-enriched Yoneda embedding. In Section 5, we describe the situation and characterise the various constructions in the case of $V = [\rightarrow, Set]$.

If $J : C_0 \longrightarrow C_1$ is a small *Freyd*-category, it is not quite true that $[J^{op}, V]$, where $V = [\rightarrow, Set]$, has a non-trivial closed *Freyd*-structure, but there is a natural factorisation of the enriched Yoneda embedding

$$C \xrightarrow{Y'} D \xrightarrow{I} [C^{op}, V]$$

with I an inclusion of a full sub-V-category (Y' is necessarily fully faithful) for which D has a canonical closed *Freyd*-structure, with Y' preserving *Freyd*-structure. In Section 6, we show that one such factorisation is characterised as the free cocompletion of C under *conical* colimits. It follows that for any small *Freyd*-category, its free conical colimit completion as a $[\rightarrow, Set]$-enriched category has a canonical closed *Freyd*-structure, and the Yoneda embedding of the *Freyd*-category into its free conical colimit completion preserves the *Freyd*-structure and yields its free conical colimit completion as a *Freyd*-category, cf the ad hoc but provably equivalent construction in [21].

Given a countable Lawvere theory, we have shown how to generate a canonical small *Freyd*-category, and given a small *Freyd*-category, we have shown how to embed it canonically into a closed *Freyd*-category. In Section 7, we consider the combination. If one adds a minor additional level of sophistication to the second construction, the combination allows us to recover the Kleisli construction for the monad on Set corresponding to the countable Lawvere theory, yielding Moggi's monad for computational effects [11,12], but now satisfying a universal condition and now with a systematic account of the operations that generate the effect.

The added sophistication is as follows: if L is a countable Lawvere theory, it follows that L^{op} has countable coproducts and the *Freyd*-structure distributes over them. One can modify our analysis of free cocompletions to account for such coproducts. Given a small category C with countable coproducts, the free cocompletion that preserves the countable coproducts may be characterised by the full subcategory $CP(C^{op}, Set)$ of $[C^{op}, Set]$ determined by countable product preserving functors from C^{op} to Set; and if C is symmetric monoidal, with tensor distributing over countable coproducts, the universality condition extends to account for symmetric monoidal structure too. That can all be modified routinely, following the work of previous sections, to the setting of *Freyd*-structure. If one does that, then starting with a countable Lawvere theory L whose induced monad is denoted T_L [19,24], the combined construction yields the closed *Freyd*-category given by the canonical functor $J : Set \longrightarrow Kl(T_L)$.

The above work impacts on the syntactic structure of programming languages. *Freyd*-categories provide a sound and complete class of models for the first-order fragment of Moggi's λ_c-calculus [9,13,16,22]. So, given a signature, its operations and equations form a countable Lawvere theory L, and L^{op} is a canonical model of the first-order fragment of the λ_c-calculus together with the signature of operations and its equations. It also models sum types and a type of natural numbers, as well as satisfying a natural universal property. Our canonical embedding of a *Freyd*-category into a closed *Freyd*-category shows how to extend that model canonically to a model of the whole λ_c-calculus, yielding a conservative extension result. Moreover, the adapted embedding respects the semantics of the sum types and the type of natural numbers. So our category theoretic analysis yields structure on the λ_c-calculus and signatures for it, as well as suggesting extensions to it. We explain this through the course of the paper as we develop our category-theoretic constructs.

Finally, we turn to recursion. Making more sophisticated use of enriched category theory again, first by enriching in the cartesian closed category ωCpo and then by allowing ωCpo to play the role of Set in the above analysis, all of the above can be modified to account for recursion, cf [3,4]. In the enriched setting, one must consider V-weighted colimits in an $[\rightarrow, V]$-category where, in the above, we considered conical colimits in an $[\rightarrow, Set]$-category, and one must replace countable products by countable cotensors. But otherwise, the above body of theory enriches without fuss, yielding an extension of the above to recursion and to effects that inherently involve recursion, such as partiality. We explain the situation in Section 8.

This paper is a journal version of parts of the conference papers [22] and [23], uniting and developing most of the main results therein. It extends the author's talk at the First Workshop on Pragmatics in Verona in 2003.

2 *Freyd*-Categories and Closed *Freyd*-Categories

In order to define the notions of *Freyd*-category and closed *Freyd*-category, we must recall the definitions of premonoidal category, strict premonoidal functor, and symmetries for them, as introduced in [25] and further studied in [21]. A premonoidal category is a generalisation of the concept of monoidal category: it is essentially a monoidal category except that the tensor need only be a functor of two variables and not necessarily be bifunctorial, i.e., given maps $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$, the evident two maps from $X \otimes X'$ to $Y \otimes Y'$ may differ.

Example 1 *Given a category C with finite products together with a specified object S, define the category K to have the same objects as C, with $K(X, Y) = C(S \times X, S \times Y)$, and with composition in K determined by that of C. For any object X of C, one has evident functors $X \otimes - : K \longrightarrow K$ and $- \otimes X : K \longrightarrow K$ extending the product in C, but they do not satisfy the bifunctoriality condition above, hence do not yield a monoidal structure on K. They do yield a premonoidal structure, as we define below.*

In order to make precise the notion of a premonoidal category, we need some auxiliary definitions.

Definition 2 *A* binoidal category *is a category K together with, for each object X of K, functors $h_X : K \longrightarrow K$ and $k_X : K \longrightarrow K$ such that for each pair (X, Y) of objects of K, $h_X Y = k_Y X$. The joint value is denoted $X \otimes Y$.*

Definition 3 *An arrow $f : X \longrightarrow X'$ in a binoidal category is* central *if for every arrow $g : Y \longrightarrow Y'$, the two composites from $X \otimes Y$ to $X' \otimes Y'$ agree. Moreover, given a binoidal category K, a natural transformation $\alpha : G \Longrightarrow H : C \longrightarrow K$ is called* central *if every component of α is central.*

Definition 4 *A* premonoidal category *is a binoidal category K together with an object I of K, and central natural isomorphisms a with components $(X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$, l with components $X \longrightarrow X \otimes I$, and r with components $X \longrightarrow I \otimes X$, subject to two equations: the pentagon expressing coherence of a, and the triangle expressing coherence of l and r with respect to a.*

Now we have the definition of a premonoidal category, it is routine to verify that Example 1 is an example of one. There is a general construction that yields premonoidal categories too: given a strong monad T on a category C with finite products, the Kleisli category $Kl(T)$ for T is always a premonoidal category, with the functor from C to $Kl(T)$ preserving premonoidal structure strictly: of course, any monoidal category, and hence any category with finite products, is trivially a premonoidal category. That construction is fundamental, albeit implicit, in Eugenio Moggi's work on monads as notions of computation [11], as explained in [25].

Definition 5 *Given a premonoidal category K, define the* centre *of K, denoted $\mathsf{Z}(K)$, to be the subcategory of K consisting of all the objects of K and* *the central morphisms.*

For an example of the centre of a premonoidal category, consider Example 1 for the case of C being the category Set. Suppose S has at least two elements. Then the centre of K is precisely Set. In general, given a strong monad on a category with finite products, the base category C need not be the centre of $Kl(T)$, but, modulo a faithfulness condition sometimes called the mono requirement [11,25], must be a subcategory of the centre.

The functors h_X and k_X preserve central maps. So we have

Proposition 6 *The centre of a premonoidal category is a monoidal category.*

This proposition allows us to prove a coherence result for premonoidal categories, directly generalising the usual coherence result for monoidal categories. Details appear in [25].

Definition 7 *A* symmetry *for a premonoidal category is a central natural isomorphism with components $c : X \otimes Y \longrightarrow Y \otimes X$, satisfying the two conditions $c^2 = 1$ and equality of the evident two maps from $(X \otimes Y) \otimes Z$ to $Z \otimes (X \otimes Y)$. A* symmetric *premonoidal category is a premonoidal category together with a symmetry.*

Example 1 is symmetric.

Definition 8 *A* strict premonoidal functor *is a functor that preserves all the structure and sends central maps to central maps.*

One may similarly generalise the definition of strict symmetric monoidal functor to strict symmetric premonoidal functor. This all allows us to define the notion of a *Freyd*-category.

Definition 9 *A Freyd*-category *is a category C_0 with finite products, a symmetric premonoidal category C_1, and an identity-on-objects strict symmetric premonoidal functor $J : C_0 \longrightarrow C_1$.*

Example 1 is one such. It follows from the definition of *Freyd*-category that every map in C_0 must be sent by J to a map in the centre $Z(C_1)$ of C_1. So it is generally safe to think of C_0 as an identity-on-objects subcategory of central maps of C_1.

Definition 10 *A Freyd-category $J : C_0 \longrightarrow C_1$ is* closed *if for every object X of C_0 (equivalently of C_1), the functor*

$$J(- \times X) : C_0 \longrightarrow C_1$$

has a right adjoint $X \to -$.

Example 1 is an example of this too if C is cartesian closed. It is proved but only stated implicitly in [25] and it is stated explicitly in [9,22] that we have:

Theorem 11 *To give a category C_0 with finite products and a strong monad on it, such that Kleisli exponentials exist, is equivalent to giving a closed Freyd-category $J : C_0 \longrightarrow C_1$.*

It follows that the class of closed *Freyd*-categories provides a sound and complete class of models for Moggi's λ_c-calculus [9,20]. Our definition of *Freyd*-category yields a definitive notion of what one might mean by the first-order fragment of the λ_c-calculus, making the class of *Freyd*-categories a sound and complete class of models for its first order fragment [9]. The details are as follows.

By the first-order fragment of the λ_c-calculus, we mean type constructors

$$\sigma ::= 1 \mid \sigma_1 \times \sigma_2$$

and term constructors

$$e ::= * \mid \langle e, e' \rangle \mid \pi_i(e) \mid let\ x = e\ in\ e' \mid x$$

where x ranges over variables, $*$ is of type 1, with π_i existing for $i = 1$ or 2, all subject to the evident typing. There are two predicates: $=$ and $(-)\downarrow$ for effect-freeness. The rules for the latter say $* \downarrow$, $x \downarrow$, if $e \downarrow$ then $\pi_i(e) \downarrow$, and similarly for $\langle e, e' \rangle$, and that definedness is closed under equality. The rules for $=$ say that $=$ is a congruence, together with rules for the basic constructions and for unit and product types. The rules are closed under substitution of effect-free terms for variables. It follows from the rules for both predicates that types together with equivalence classes of terms in context form a category, with a subcategory determined by effect-free terms.

The *let* constructor is derivable in the full λ_c-calculus as $(\lambda x.e')e$. It follows from our construction that the class of *Freyd*-categories provides a sound and complete class of models for the first-order fragment of the λ_c-calculus just as that of closed *Freyd*-categories provides a sound and complete class of models for the full calculus.

3 From Countable Lawvere Theories to ***Freyd***-categories

It is generally clear, given a computational effect, how to choose suitable operations that generate it. For instance, in modelling nondeterminism, one typically starts with binary $\vee$; for global state, one typically chooses *lookup* and *update*; and for interactive input/output, one considers *read* and *write*. It is less clear what equations to impose as axioms, and that question deserves systematic treatment, cf [23]. But in particular cases, such as the above, there are generally agreed computationally natural equations: for nondeterminism, one demands associativity, commutativity and idempotence; for global state, one demands the equations listed in Example 13; and for interactive input/output, one typically demands no equations [14].

Equations typically hold between derived operations rather than between primitive ones. For instance, to express associativity of $\vee$, one must be able to speak of $(x \vee y) \vee z$, which is given by a derived ternary operation. So, we seek a unified way in which to speak of the derived operations generated by a signature. There are several equivalent ways to do that, and we shall use the notion of *countable Lawvere theory* [3].

Let $\aleph_1$ denote a skeleton of the category of countable sets and all functions between them. So $\aleph_1$ has an object for each natural number n and an object for $\aleph_0$. Up to equivalence, $\aleph_1$ is the free category with countable coproducts on 1. So, in referring to $\aleph_1$, we implicitly make a choice of the structure of its countable coproducts.

Definition 12 *A* countable Lawvere theory *is a small category L with countable products and a strict countable-product preserving identity-on-objects functor $I : \aleph_1^{op} \longrightarrow L$.*

Implicit in the definition is the statement that $\aleph_1^{op}$ and L have the same set of objects. We typically write L for a countable Lawvere theory, with the data given by $I : \aleph_1^{op} \longrightarrow L$ left implicit. Every signature of operations, with arities either natural numbers or $\aleph_0$, subject to universally defined equations, freely generates a countable Lawvere theory. The arrows with domain n and codomain 1 in that countable Lawvere theory are exactly the equivalence classes of derived n-ary operations generated by the signature; an arrow with domain n and codomain m consists exactly of m equivalence classes of derived n-ary operations generated by the signature. And that generalises routinely to $\aleph_0$. Composition in the countable Lawvere theory amounts to a category theoretic formulation of the notion of substitution.

Example 13 *A signature for global state contains operations $lookup : Val \longrightarrow Loc$ and $update : 1 \longrightarrow Loc \times Val$, where Loc is a finite set of locations and Val is a countable set of values [4,16]. These freely generate a countable Lawvere* *theory by identifying the finite set Loc with its cardinality n and by identifying Val with $\aleph_0$, then freely allowing substitutions applied to instances of lookup and update. So an arrow is a word of finite length but possibly infinite breadth of copies of lookup and update. These operations are now subject to seven equation schema, which, with lookup corresponding to the logical symbol l and with update corresponding to u, can be expressed syntactically as*

(1) $l_{loc}(u_{loc,v}(x))_v = x$
(2) $l_{loc}(l_{loc}(t_{vv'})_v)_{v'} = l_{loc}(t_{vv})_v$
(3) $u_{loc,v}(u_{loc,v'}(x)) = u_{loc,v'}(x)$
(4) $u_{loc,v}(l_{loc}(t_{v'})_{v'}) = u_{loc,v}(t_v)$
(5) $l_{loc}(l_{loc'}(t_{vv'})_{v'})_v = l_{loc'}(l_{loc}(t_{vv'})_v)_{v'}$ *where* $loc \neq loc'$
(6) $u_{loc,v}(u_{loc',v'}(x)) = u_{loc',v'}(u_{loc,v}(x))$ *where* $loc \neq loc'$
(7) $u_{loc,v}(l_{loc'}(t_{v'})_{v'}) = l_{loc'}(u_{loc,v}(t_{v'}))_{v'}$ *where* $loc \neq loc'$.

The countable Lawvere theory given by factoring out by these equations is the countable Lawvere theory L_S for global state.

Example 14 *A signature for interactive input/output consists of operations $read : I \longrightarrow 1$ and $write : 1 \longrightarrow O$, for countable sets I of inputs and O of outputs [16,4]. Again, identifying I and O with $\aleph_0$, these operations freely generate a countable Lawvere theory that we call the countable Lawvere theory $L_{I/O}$ for interactive input/output.*

Exceptions work much as interactive input/output: the countable Lawvere theory L_E is freely generated by an operation $raise : 0 \longrightarrow E$ for a countable set of exceptions E [16,4]. Nondeterminism involves issues of partiality that we do not treat in this section, but the heart of it is given by the free countable Lawvere theory L_N on a binary operation $\vee$ subject to equations for associativity, commutativity, and idempotence [16]. Of course, one can also consider combinations of such effects [3,4].

Trivially, to give the strict countable-product preserving functor $I : \aleph_1^{op} \longrightarrow L$ in the definition of a countable Lawvere theory is equivalent to giving a strict countable-coproduct preserving functor $J : \aleph_1 \longrightarrow L^{op}$. The category $\aleph_1$ not only has countable coproducts but also has finite products: these are given by finite products of countable sets. The category L^{op} generally does not have finite products, and the finite products of $\aleph_1$ are generally not preserved by J. But one can routinely check the following result:

Theorem 15 *For any countable Lawvere theory L, the category L^{op} together with the functor $I^{op} : \aleph_1 \longrightarrow L^{op}$ canonically support the structure of a Freyd-category.*

PROOF. Given a countable (possibly finite) set α and given a map in L, say $f : \beta \longrightarrow \gamma$, we must define a map $\alpha \otimes f$ in L from $\alpha \times \beta$ to $\alpha \times \gamma$. The set $\alpha \times \beta$ is the sum of α-many copies of β, and similarly for $\alpha \times \gamma$. The category L^{op} has countable sums, and countable sums are preserved by I^{op}. So we define $\alpha \otimes f : \alpha \times \beta \longrightarrow \alpha \times \gamma$ to be the sum in L^{op} of α copies of f: the domain and codomain of this sum are as desired because I^{op} preserves countable sums. This determines the rest of the data for a *Freyd*-structure, and it is routine to verify that the *Freyd*-category axioms all hold.

This allows us to extend our analysis of the first-order fragment of the λ_c-calculus at the end of Section 2 as follows.

Corollary 16 *For any countable Lawvere theory L, the category L^{op} together with $I^{op} : \aleph_1 \longrightarrow L^{op}$ is a model of the first-order fragment of the λ_c-calculus.*

We call the countable Lawvere theory of Corollary 16 the *canonical* model determined by the computational effect associated with L.

Next consider *exactly* what one might mean by an interpretation of the operations of a signature for the first-order fragment of the λ_c-calculus. In previous work, we have investigated three main ways to interpret operations in the setting of the full λ_c-calculus [15]. When considered in the context of a closed *Freyd*-category, all three are equivalent. But in the absence of closedness, we can define only two of those notions of interpretation; they remain equivalent to each other. The difficulty for the third notion arises because when S is countable, $S \to (X \times S)$ is uncountable even when $X = 1$ [15]. Here, we focus on the notion that most directly yields a canonicity result. It uses the idea of a *generic effect*.

Definition 17 *Given a signature of typed basic operations and given a semantics for each type, an* interpretation *of an operation of type $\sigma \to \tau$ in a Freyd-category $J : C_0 \longrightarrow C_1$ is a map $M(\tau) \longrightarrow M(\sigma)$ in C_1, where $M(\sigma)$ and $M(\tau)$ are the interpretations of the types σ and τ.*

Example 18 *Consider the usual interpretation of side-effects in the Kleisli category $Kl(S \to (- \times S))$ for the monad $S \to (- \times S)$ on Set, where $S = Val^{Loc}$. The operation lookup : $Val \longrightarrow Loc$ is interpreted by the function*

$$Loc \longrightarrow (S \to (Val \times S))$$

taking (loc, σ) to (v, σ), where v is given by looking up loc in σ. To give a function from Loc to $(S \to (Val \times S))$ is to give a map in $Kl(S \to (- \times S))$ from Loc to Val. The operation update : $1 \longrightarrow Loc \times Val$ is interpreted by the function

$$Loc \times Val \longrightarrow (S \to S)$$

sending (loc, v, σ) to the state that updates σ by replacing the value at loc by v; and that is a map in $Kl(S \to (- \times S))$ from $Loc \times Val$ to 1. This way of modelling operations as generic effects *has proved particularly useful [15,3,4] and is consistent with Example 13 here. If we restrict from the λ_c-calculus to its first-order fragment, we can restrict the interpretation to land in the full sub-Freyd-category of $Kl(S \to (- \times S))$ determined by (a skeleton of) countable sets. This latter Freyd-category is exactly the canonical Freyd-category for global state determined by Corollary 16. It is not yet clear how to incorporate local state into the setting of this paper, although there is reason for optimism that that will be possible in due course [14].*

One can similarly use the notion of interpretation as we have defined it here to give canonical interpretations of $\vee$ for nondeterminism, *read* and *write* for interactive input/output, *raise* for exceptions, etcetera [15], all respecting the appropriate equations. One has the following trivial but fundamental proposition:

Proposition 19 *Every signature of operations of countable arity has a canonical sound interpretation in the canonical model: an arity α is modelled by the object α, and a basic operation $op : \alpha \longrightarrow \beta$ is modelled by the corresponding map from β to α in L^{op}.*

4 Enrichment in $[\to, Set]$

In this section, we describe enriched categories, in particular with respect to enrichment in $[\to, Set]$, and we characterise the latter. The standard reference for enriched categories is [6]. For simplicity of exposition, we shall restrict our attention to enrichment in a complete and cocomplete cartesian (rather than just monoidal or symmetric monoidal) closed category V.

Definition 20 *A V-category C consists of*

- *a set $Ob(C)$ of* objects
- *for every pair (X, Y) of objects of C, an object $C(X, Y)$ of V*
- *for every object X of C, a map $\iota : 1 \longrightarrow C(X, X)$*
- *for every triple (X, Y, Z), a map*

$$\cdot : C(Y, Z) \times C(X, Y) \longrightarrow C(X, Z)$$

subject to an associativity axiom for $\cdot$ and an axiom making ι a left and right unit for $\cdot$.

The leading example has $V = Set$, in which case the notion of V-category agrees exactly with the usual notion of locally small category. Other standard examples involve $V = Poset$, yielding locally small locally ordered categories, $V = \omega Cpo$, yielding locally small categories with coherent ωcpo structure on each homset, allowing an account of recursion, and $V = Cat$, yielding locally small 2-categories. But the example of primary interest to us here has $V = [\rightarrow, Set]$: the category $\rightarrow$ is the category determined by a pair of objects and one non-identity arrow, which goes from the first object to the second; so an object of the functor category $[\rightarrow, Set]$ consists of a pair of sets (X_0, X_1) and a function from one to the other, $f : X_0 \longrightarrow X_1$, and an arrow amounts to a commutative square in Set. Products are given pointwise; the closed structure is more complicated, cf Proposition 23.

Proposition 21 *To give an $[\rightarrow, Set]$-category is equivalent to giving a pair of categories and an identity-on-objects functor $J : C_0 \longrightarrow C_1$.*

PROOF. Given an $[\rightarrow, Set]$-category C, put $Ob(C_0) = Ob(C_1) = Ob(C)$. For any pair (X, Y) of objects of C, the data for an $[\rightarrow, Set]$-category gives us an object $C(X, Y)$ of $[\rightarrow, Set]$, i.e., a pair of sets and a function $f : A \longrightarrow B$. So *define* $C_0(X, Y) = A$ and $C_1(X, Y) = B$, and *define* the behaviour of the putative functor $J : C_0 \longrightarrow C_1$ on the homset $C_0(X, Y)$ to be $f : C_0(X, Y) \longrightarrow C_1(X, Y)$. The rest of the data and the axioms for an $[\rightarrow, Set]$-category provide the rest of the data and axioms to make C_0 and C_1 into categories and to make J functorial. The converse follows by similarly routine calculation.

Based on this result, we henceforth identity the notion of $[\rightarrow, Set]$-category with a pair of categories and an identity-on-objects functor $J : C_0 \longrightarrow C_1$.

In general, every V-category C has an underlying ordinary category $U(C)$ defined by $Ob(U(C)) = Ob(C)$ and with the homset $(UC)(X, Y)$ defined to be the set of maps in V from the terminal object 1 to $C(X, Y)$. The composition of the V-category C routinely induces a composition for $U(C)$, and similarly for the identity maps.

Proposition 22 *The underlying ordinary category of an $[\rightarrow, Set]$-category $J : C_0 \longrightarrow C_1$ is the category C_0.*

This result follows from routine checking.

5 The $[\rightarrow, Set]$-enriched Yoneda Embedding

The Yoneda embedding $Y : C \longrightarrow [C^{op}, Set]$ has a subtle universal property: it is the free colimit completion, or more briefly the free cocompletion, of a small category C [6]. Moreover, if C has finite products, it is the free finite product cocompletion of C, and if C is symmetric monoidal, it is the free symmetric monoidal cocompletion of C [5]. We shall give a variant of this universal property for *Freyd*-categories in Section 6, but in order to do so, we need to study the enriched Yoneda embedding $Y : C \longrightarrow [C^{op}, V]$ in the setting of $V = [\rightarrow, Set]$. To do that, we first observe that V itself has the structure of a V-category, with homobject $V(X,Y)$ given by the exponential Y^X of V. This yields the following result in the case of $V = [\rightarrow, Set]$:

Proposition 23 *The cartesian closed category $[\rightarrow, Set]$ extends canonically to the $[\rightarrow, Set]$-category*

$$inc : [\rightarrow, Set] \longrightarrow [\rightarrow, Set]_1$$

where $[\rightarrow, Set]_1(f : X \longrightarrow Y, f' : X' \longrightarrow Y')$ is defined to be the set of functions from Y to Y'. The behaviour of the functor inc is evident.

Observe, prefiguring a deeper use of this idea we shall make later, that the category $[\rightarrow, Set]_1$ and the functor *inc* are given by the identity-on-objects/fully faithful factorisation of the codomain functor from $[\rightarrow, Set]$ to Set.

For any small V-category C, one has a functor V-category $[C, V]$. In general, given V-categories C and D, a V-functor $H : C \longrightarrow D$ consists of a function $H_{Ob} : Ob(C) \longrightarrow Ob(D)$ together with, for each pair of objects (XY) of C, a map $C(X,Y) \longrightarrow D(HX, HY)$ in V, subject to two axioms to the effect that composition and identities are respected. This is a routine generalisation of the usual notion of functor. An object of $[C, V]$ is a V-functor from C to V and the homobject $[C, V](H, K)$ is given by an equaliser that internalises to V the construction of the set of natural transformations between parallel functors: details appear in [6] but we now spell out the situation in the case of $V = [\rightarrow, Set]$.

Proposition 24 *Given a small $[\rightarrow, Set]$-category $J : C_0 \longrightarrow C_1$, the functor $[\rightarrow, Set]$-category $[J, [\rightarrow, Set]]$ is defined as follows: an object consists of*

- *a functor $H_0 : C_0 \longrightarrow Set$*
- *a functor $H_1 : C_1 \longrightarrow Set$*
- *a natural transformation $\phi : H_0 \Rightarrow H_1 J$*

$$\begin{tikzcd} C_0 \arrow[r, "J"] \arrow[dr, "H_0"'] & C_1 \arrow[d, "H_1"] \arrow[dl, phantom, "\Uparrow \phi"] \\ & Set \end{tikzcd}$$

An arrow in $[J, [\rightarrow, Set]]_0$ from (H_0, K_0, ϕ_0) to (H_1, K_1, ϕ_1) consists of a pair *of natural transformations ($H_0 \Rightarrow K_0, H_1 \Rightarrow K_1$) making the evident diagram involving the ϕ's commute. An arrow in $[J, [\rightarrow, Set]]_1$ between the same objects consists of a natural transformation $H_1 \Rightarrow K_1$. Composition and the behaviour of the identity-on-objects functor are evident.*

PROOF. This follows by consideration of the definition of the V-category $[C, V]$ where $V = [\rightarrow, Set]$. An object of $[C, V]$ in this setting consists of an $[\rightarrow, Set]$-functor from J to $[\rightarrow, Set]$ regarded as a $[\rightarrow, Set]$-category using the construction of Proposition 23. Such a functor assigns, to each object X of C_0, equivalently each object X of C_1, an arrow in Set, giving precisely the data for the object parts of H_0 and H_1 and the natural transformation ϕ. The behaviour of the $[\rightarrow, Set]$-functor on homs is equivalent to the behaviour of H_0 and H_1 on arrows. And the various axioms for a $[\rightarrow, Set]$-functor are equivalent to functoriality of H_0 and H_1 and naturality of ϕ. Similarly routine calculations yield the characterisations of the two sorts of arrow in the functor $[\rightarrow, Set]$-category.

We can further characterise this $[\rightarrow, Set]$-category by means of a lax colimit in the 2-category Cat [1].

Definition 25 *Given a functor $J : C_o \longrightarrow C_1$, denote by $l(J)$ the category determined by being universal of the form*

$$\begin{tikzcd} C_0 \arrow[r, "J"] \arrow[dr, "I_0"'] & C_1 \arrow[d, "I_1"] \arrow[dl, phantom, "\Uparrow \iota"] \\ & l(J) \end{tikzcd}$$

I.e., for every such diagram with an arbitrary vertex D, there is a unique functor from $l(J)$ to D making corresponding functors and natural transformations agree.

One can provably extend the condition of the definition uniquely to yield an isomorphism of categories between a category with such lax cocones with vertex D as objects and the functor category from $l(J)$ to D. It is easy to construct $l(J)$: it is freely generated by having C_0 and C_1 as full subcategories, together with, for each object X of C_0, an arrow from $I_0(X)$ to $I_1(X)$, subject to the collection of such arrows being made natural in C_0. Note that the coprojections I_0 and I_1 are fully faithful. The universal property tells us that to give an object of $[J, [\rightarrow, Set]]$ is equivalent to giving a functor from $l(J)$ to Set, allowing us to deduce the following proposition.

Proposition 26 *The functor $[\to, Set]$-category $[J, [\to, Set]]$ is given by the identity-on-objects/fully faithful factorisation of*

$$[I_1, Set] : [l(J), Set] \longrightarrow [C_1, Set]$$

i.e., $[J, [\to, Set]]_0$ is isomorphic to $[l(J), Set]$, and $[J, [\to, Set]]_1$ and the identity-on-objects functor are given by the identity-on-objects/fully faithful factorisation of $[I_1, Set]$.

Finally, we investigate the enriched Yoneda embedding $Y : C \longrightarrow [C^{op}, V]$ when $V = [\to, Set]$. Given an $[\to, Set]$-category $J : C_0 \longrightarrow C_1$, the Yoneda embedding consists of an $[\to, Set]$-functor from J to $[J^{op}, [\to, Set]]$, i.e., a pair of functors

$$(Y_0 : C_0 \longrightarrow [J^{op}, [\to, Set]]_0, Y_1 : C_1 \longrightarrow [J^{op}, [\to, Set]]_1)$$

We can characterise these functors as follows:

Proposition 27 *The functor $Y_0 : C_0 \longrightarrow [J^{op}, [\to, Set]]_0$ is the composite of the ordinary Yoneda embedding $Y : C_0 \longrightarrow [C_0^{op}, Set]$ with the (fully faithful) functor $Lan_{I_0} : [C_0^{op}, Set] \longrightarrow [l(J^{op}), Set] = [J^{op}, [\to, Set]]_0$. And the functor $Y_1 : C_1 \longrightarrow [J^{op}, [\to, Set]]_1$ is given by the defining property of a factorisation system applied to the square*

$$\begin{tikzcd}
C_0 \arrow[r, "J"] \arrow[d] & C_1 \arrow[d, "Y"] \\
{[J^{op}, [\to, Set]]_1} \arrow[r] & {[C_1^{op}, Set]}
\end{tikzcd}$$

where the bottom (fully faithful) functor is given by applying $(-)^{op}$ to Proposition 26 and the left-hand functor is given by the composite of Y_0 with the identity-on-objects functor determined by applying $(-)^{op}$ to Proposition 26. The definition of a factorisation system (this can also be proved directly) yields a unique functor from C_1 to $[J^{op}, [\to, Set]]_1$ making both triangles commute.

PROOF. The enriched Yoneda embedding takes an object X of C_0, equivalently of C_1, to the $[\to, Set]$-functor $J(-, X) : J^{op} \longrightarrow [\to, Set]$, which may be described as the pair of functors

$$(C_0(-, X) : C^{op} \longrightarrow Set, C_1(-, X) : C_1^{op} \longrightarrow Set)$$

together with the natural transformation from the first to the second determined by J. It is routine to verify that $C_1(-,X) : C_1^{op} \longrightarrow Set$ is the left Kan extension of $C_0(-,X)$ along J^{op} (see [6] or [10] for the definition and properties of Kan extensions). And by composition of left Kan extensions, its left Kan extension along I_1 agrees with $Lan_{I_0}C_0(-,X)$. Since I_0 and I_1 are fully faithful, it follows that $Lan_{I_0}C_0(-,X)$ commutes with both $C_0(-,X)$ and $C_1(-,X)$, respecting ι. This proves the characterisation we claim for Y_0, and that for Y_1 follows because, by its definition, it must be the unique functor making the two triangles commute.

The behaviour of the two functors on maps follows routinely if we can see that Lan_{I_0} and Lan_{I_1} are fully faithful. The proof is the same for both, so let us just consider I_0. Since I_0 is fully faithful, it follows (see for instance [6]) as used above that for any functor $H : C_0^{op} \longrightarrow Set$, we have that H_0 is coherently isomorphic to the composite $(Lan_{I_0}H_0)I_0$. But $Lan_{I_0} : [C_0^{op}, Set] \longrightarrow [l(J^{op}), Set]$ has a right adjoint given by sending a functor to its composite with I_0, and the above-mentioned isomorphism tells us that the unit of the adjunction is an isomorphism, and hence that the adjunction is a coreflection, and hence that Lan_{I_0} is fully faithful.

6 The Free Conical Colimit Completion of a Small $[\rightarrow, Set]$-Category

Weighted colimits, sometimes called *indexed* colimits, form the definitive notion of colimit in an enriched category [6]. But the definition is complex and we do not need it in this paper except to study recursion later. *Conical* colimits, which amount to the first obvious guess for a notion of enriched colimit, are among the weighted colimits but are not all of them. Moreover, they are exactly the colimits we need in our analysis of $V = [\rightarrow, Set]$. If V were Set, the small conical colimit completion of a small V-category C would be exactly $[C^{op}, Set]$, but that is not true for general V, and in particular, it is not true for $V = [\rightarrow, Set]$. So, in this section, we describe conical colimits and characterise the conical colimit completion of a small V-category in the setting where $V = [\rightarrow, Set]$. We then use that construction to give a canonical embedding of a small *Freyd*-category into a closed *Freyd*-category.

Given a V-category C and a small ordinary category L, one can construct a V-category $[L, C]$. An object of $[L, C]$ is a functor from L to $U(C)$. Given functors $H, K : L \longrightarrow U(C)$, one defines the homobject $[L, C](H, K)$ of V to be an equaliser in V of two maps of the form

$$\Pi_{Ob(L)}C(HX, KX) \longrightarrow \Pi_{ArrL}C(HX, KY)$$

one determined by postcomposition with Kf, the other given by precomposition with Hf, for each map f in L, thus internalising the notion of natural transformation. When $V = Set$, this construction agrees with the usual definition of the functor category.

Definition 28 *For an arbitrary V-category C and a small ordinary category L, given a functor $H : L \longrightarrow U(C)$, a* conical colimit *of H is a cocone over H with vertex defined to be $colimH$, such that composition with the cocone yields, for every object X of C, an isomorphism in V of the form*

$$C(colimH, X) \cong [L, C](H, \Delta X)$$

If $V = Set$, this definition agrees with the usual notion of colimit. In general, a V-category C is said to have all conical colimits if, for every small category L, every functor $H : L \longrightarrow U(C)$ has a conical colimit.

Theorem 29 *[6] The free conical colimit completion of a small V-category C is given by the closure of C in $[C^{op}, V]$ with respect to the Yoneda embedding $Y : C \longrightarrow [C^{op}, V]$ under conical colimits.*

We proceed to characterise the construction of Theorem 29 in the case of $V = [\rightarrow, Set]$.

Proposition 30 *An $[\rightarrow, Set]$-category $J : C_0 \longrightarrow C_1$ has all conical colimits if and only if C_0 has all colimits and J preserves all colimits.*

PROOF. Suppose the $[\rightarrow, Set]$-category $J : C_0 \longrightarrow C_1$ has all conical colimits. In general, if an arbitrary V-category has all conical colimits, it follows that its underlying ordinary category has all colimits. So C_0 has all colimits. Now, by direct use of the definition of conical colimits in the case of $V = [\rightarrow, Set]$, it follows that J must preserve them. The converse holds by direct calculation.

Theorem 31 *The free conical colimit completion of a small $[\rightarrow, Set]$-category $J : C_0 \longrightarrow C_1$ is given by*

- *the category $[C_0^{op}, Set]$*
- *the identity-on-objects/fully faithful factorisation of*

$$Lan_{J^{op}} : [C_0^{op}, Set] \longrightarrow [C_1^{op}, Set]$$

PROOF. First observe, using Proposition 30, that this $[\rightarrow, Set]$-category has conical colimits: $[C_0^{op}, Set]$ is cocomplete and $Lan_{J^{op}}$ has a right adjoint, and factoring a colimit preserving functor into an identity-on-objects functor followed by a fully faithful functor makes the former also preserve all colimits. Now observe that the canonical $[\rightarrow, Set]$-functor into $[J^{op}, [\rightarrow, Set]]$ preserves colimits: the canonical $[\to, Set]$-functor is given by Lan_{I_0}, which has a right adjoint, so preserves colimits, together with the functor determined by the universal property of a factorisation system applied to the commutative square

$$
\begin{tikzcd}
{[C_0^{op}, Set]} \arrow[r] \arrow[d, "Lan_{I_0}"'] & C' \arrow[r] & {[C_1^{op}, Set]} \arrow[d, "id"] \\
{[J^{op}, [\to, Set]]_0} \arrow[r] & {[J^{op}, [\to, Set]]_1} \arrow[r] & {[C_1^{op}, Set]}
\end{tikzcd}
$$

where the top and bottom rows are given by identity-on-objects/fully faithful factorisations, and where the diagram commutes by calculations with left Kan extensions and using fully faithfulness of I_1. The canonical $[\to, Set]$-functor is fully faithful: we established fully faithfulness of Lan_{I_0} in the previous section, and the intermediary functor as above is fully faithful as its composite with the bottom right-hand functor in the diagram is fully faithful. Next observe, by Proposition 27, that the Yoneda embedding factors through the canonical $[\to, Set]$-functor. Finally, observe that every object of this full sub-$[\to, Set]$-category is generated by a conical colimit of representables: that is routine as every functor $H : C_0^{op} \longrightarrow Set$ is a conical colimit of representables [6]. Combining all these observations yields the result.

Turning now to *Freyd*-structure, in general, for a small V-category C, the functor V-category $[C^{op}, V]$ has finite products, indeed all limits and colimits. We have characterised conical colimits in the setting of $V = [\to, Set]$ in Proposition 30. A dual result applies to conical limits. So, for any small $[\to, Set]$-category $J : C_0 \longrightarrow C_1$, it follows that in the presheaf $[\to, Set]$-category

$$[J^{op}, [\to, Set]]_0 \longrightarrow [J^{op}, [\to, Set]]_1$$

the category $[J^{op}, [\to, Set]]_0$ has and the functor preserves finite products. So it does not have non-trivial *Freyd*-structure. So, in particular, it does not provide a closed *Freyd*-category into which J, equipped with a non-trivial *Freyd*-structure, can embed as a *Freyd*-category. We therefore cannot adapt the construction of the free finite product cocompletion of a category with finite products to the setting of *Freyd*-structure simply by enrichment of the Yoneda embedding in $V = [\to, Set]$: we *must* add further subtlety. That subtlety is given by restricting the Yoneda embedding to the conical colimit completion, and it agrees with the ad hoc description of an embedding of (something very similar to) a small *Freyd*-category into (something very similar to) a closed *Freyd*-category in [21]: from Theorem 31, it is little more than an observation that the proof in [21] extends to yield a universal characterisation of the construction therein.

There is one delicate point: exactly what do we mean by the "free conical-colimit complete closed *Freyd*-category on a *Freyd*-category?" By a map of closed *Freyd*-categories, we mean a map that preserves the *Freyd*-structure but need not preserve the closed structure. This should not come as a great surprise: the maps of primary interest between cartesian closed categories are functors that preserve finite products but need not preserve the closed structure; most forgetful functors to Set are examples. It also agrees with the universal characterisations for finite product and symmetric monoidal structure in [5] and with the work of [8] on data refinement.

Theorem 32 *The free conical colimit completion of a small Freyd-category $J : C_0 \longrightarrow C_1$ is the free conical-colimit complete closed Freyd-category on J, i.e., the $[\rightarrow, Set]$-category of Theorem 31 with a natural Freyd-structure.*

PROOF. Theorem 29 characterises the free conical colimit completion of any $[\rightarrow, Set]$-category $J : C_0 \longrightarrow C_1$. We need only show that that construction acts as we wish with respect to *Freyd*-structure. But $[C_0^{op}, Set]$ is cartesian closed, with $Y : C_0 \longrightarrow [C^{op}, Set]$ preserving finite products, and $Lan_{J^{op}}$ has a right adjoint. So the only remaining non-trivial point is to construct, for functors $F, H, K : C_0^{op} \longrightarrow Set$ and for every natural transformation $\alpha : Lan_{J^{op}}H \Rightarrow Lan_{J^{op}}K$, a natural transformation $Lan_{J^{op}}(F \times H) \Rightarrow Lan_{J^{op}}(F \times K)$; and that must be done coherently. But to do that, we just make two uses of the fact that $[C^{op}, Set]$ is the free colimit completion of C. It follows from this free cocompleteness that, for any object X of C_0, equivalently of C_1, the functor $X \otimes - : C_1 \longrightarrow C_1$ extends to $[C_1^{op}, Set]$. This yields $F \otimes \alpha$ for any representable $F = C_0(-, X)$. For an arbitrary F, one deduces the construction by use of symmetry and by centrality of the maps in the canonical colimiting cocone of F.

7 Recovering Monads

In this section, we consider the composite of our two constructions: the first building a *Freyd*-category from a countable Lawvere theory or, in practice, from a signature of operations and equations, the second building a closed *Freyd*-category from a *Freyd*-category. If we refine Theorem 29 a little, following the work in [6], we recover Moggi's monads for computational effects [11,12]. In order to show how this works, we start with a general theorem about enriched categories [6]. We then study what that theorem says in the case of $V = [\rightarrow, Set]$ and see how it applies to *Freyd*-structure. The general theorem is as follows.

Theorem 33 *If C is a small V-category with countable coproducts, the free* *conical colimit completion of C that preserves the countable coproducts of C is given by the closure of C in $CP(C^{op}, V)$ with respect to the Yoneda embedding under conical colimits.*

A priori, this result is relevant to us because any countable Lawvere theory L has countable products and so L^{op} has countable coproducts. That is essentially the information we use, but we need a slightly more subtle use of it as we need to consider $V = [\rightarrow, Set]$. One can further adapt Theorem 32 along the same lines as Theorem 33 is an adaptation of Theorem 29, cf [21,25]. We proceed as follows.

Definition 34 *A Freyd-category $J : C_0 \longrightarrow C_1$ has* countable coproducts *if C_0 has and J preserves countable coproducts.*

Proposition 35 *For any closed Freyd-category $J : C_0 \longrightarrow C_1$, if C_0 has countable coproducts, so does J.*

There is more flexibility here than might first appear. If a cartesian closed category C has countable coproducts, it follows that, for every object X of C, the functor $- \times X : C \longrightarrow C$ preserves them, i.e., product distributes over sum. But if C has finite products and countable coproducts without being closed, $- \times X$ might not preserve countable coproducts. But many categories do satisfy such a preservation condition and it is remarkably powerful, yielding the notion of a countably *distributive* category [2]. The same issue arises for *Freyd*-categories: in the presence of countable coproducts, one is naturally led to the notion of countably distributive *Freyd*-category, implying axioms on an extension of the λ_c-calculus to include sum types as we shall discuss shortly. Here, we have and need a notion of countable distributivity anyway.

Definition 36 *A Freyd-category $J : C_0 \longrightarrow C_1$ is* countably distributive *if C_0 has and J strictly preserves countable coproducts, and finite products distribute over countable coproducts in C_0.*

The notion of countable distributivity allows us to characterise the canonical model of Corollary 16 by a universal property.

Theorem 37 *The canonical model is the generic countably distributive Freyd-category, i.e., for any countably distributive Freyd-category $J : C_0 \longrightarrow C_1$ and any sound interpretation of the signature in J that respects the coproduct structure of the arities, there is, up to coherent isomorphism, a unique countable coproduct preserving Freyd-functor from I^{op} to J that respects the interpretations.*

Now we can state the result we really want.

Theorem 38 *The free conical colimit completion of a small countably dis-*

tributive Freyd-category $J : C_0 \longrightarrow C_1$ that preserves the countable coproducts of J is the free conical-colimit complete closed Freyd-category on J that preserves the countable coproducts of J, i.e, the $[\rightarrow, Set]$-category of Theorem 33 in the case of $V = [\rightarrow, Set]$ and taking C to be J, with a natural Freyd-structure.

The construction of the closed *Freyd*-structure is exactly as in Theorem 32 except for the systematic replacement of arbitrary functors by ones that respect the countable coproduct structure of C.

It follows from the definition of countable Lawvere theory that if L is a countable Lawvere theory, the *Freyd*-category $I^{op} : \aleph_1 \longrightarrow L^{op}$ is countably distributive. So, starting with a countable Lawvere theory, then applying Theorem 15 followed by Theorem 38, we obtain the identity-on-objects/fully faithful factorisation of a functor of the form

$$CP(\aleph_1^{op}, Set) \longrightarrow CP(L, Set)$$

But $\aleph_1^{op}$ is the free category with countable products on 1. So the category $CP(\aleph_1^{op}, Set)$ is equivalent to Set, and thus we have the identity-on-objects/fully faithful factorisation of a functor of the form

$$Set \longrightarrow CP(L, Set)$$

It is a standard result of Lawvere theories that the category $CP(L, Set)$ is monadic over Set with monad T_L induced by L (see [19,24]). So having routinely checked some coherence details, we have the identity-on-objects/fully faithful factorisation of the canonical left adjoint

$$Set \longrightarrow T_L - Alg$$

and that factorisation yields precisely $Kl(T_L)$. Thus we have the following:

Theorem 39 *For any countable Lawvere theory $J : \aleph_1^{op} \longrightarrow L$, the canonical closed Freyd-category*

$$Set \longrightarrow Kl(T_L)$$

is the free conical colimit completion of $I^{op} : \aleph_1 \longrightarrow L^{op}$ that preserves the countable coproducts of I^{op}.

For calculi, the canonical model of Corollary 16 agrees and unifies the models for the various computational effects given by Moggi: he did not give a unified way to model signatures, so the best we can do is to point out that our unified account agrees with all his examples.

The result means our analysis decomposes the construction of the Kleisli category for a monad into two parts whenever the monad arises from a countable Lawvere theory. In all the examples of computational effects we address here, that is the case, and so this decomposition refines Moggi's analysis, adds a systematic account of operations, and allows one a more structured development of the associated λ_c-calculus.

Our work also suggests an extension of the first-order fragment of the λ_c-calculus to include sum types. The canonical model is a finitely distributive *Freyd*-category. So, by the first-order fragment of the λ_c-calculus with sum types, we might mean type constructors

$$\sigma ::= 1 \mid \sigma_1 \times \sigma_2 \mid 0 \mid \sigma_1 + \sigma_2$$

and term constructors

$$e ::= * \mid \langle e, e' \rangle \mid \pi_i(e) \mid let\ x = e\ in\ e' \mid 0 \mid inl(e) \mid inr(e) \mid cases(e_1, e_2) \mid x$$

subject to evident typing rules and an extension of the rules for the predicates $=$ and $(-)\downarrow$ to make the class of finitely distributive *Freyd*-categories $J : C_0 \longrightarrow C_1$ into a sound and complete class of models.

One typically does not have countable sum types directly in an idealised programming language such as the λ_c-calculus, but one does typically have Nat, and that is also canonically modelled in the canonical model generated by any signature. Data and axioms for Nat are already definitive, so our work here does not yield new insight there, but at least it is consistent.

8 Recursion through Enrichment

Recursion may added to a study of computational effects in the spirit of the above work systematically by changing base category from Set to ωCpo, e.g., as in [4], changing from ordinary functors to ωCpo-enriched functors, etcetera. Many of the constructions of ordinary category theory enrich without fuss; but a few, especially those involving limits, require greater care, in particular because products in the definition of Lawvere theory enrich most naturally as *cotensors* [6,4,19]. With care, all of the category theoretic work of the paper does generalise to enrichment in a cartesian closed category V satisfying standard axiomatic conditions. In this section, we outline how the enrichment works.

Assume V is locally countably presentable as a cartesian closed category [7]: one does not need a formal definition to follow the work of this section; the main point is that it includes categories such as ωCpo and $Poset$. If we systematically add enrichment to the definitions associated with the notion of premonoidal category, we can make the following definition, enriching Definitions 9 and 10.

Definition 40 *A Freyd-V-category is a V-category C_0 with finite products, a small symmetric premonoidal V-category C_1, and an identity-on-objects strict symmetric premonoidal V-functor $J : C_0 \longrightarrow C_1$. It is* closed *if for every object X of C_0, the V-functor $J(- \times X) : C_0 \longrightarrow C_1$ has a right V-adjoint.*

Letting $V_{\aleph_1}$ be a skeleton of the full sub-V-category of V determined by countably presentable objects of V, we can define the notion of a countable Lawvere V-theory [4]. Given an object X of V and an object A of a V-category C, an X-*cotensor* of A is an object A^X of C for which there is an isomorphism

$$C(B, A^X) \cong C(B, A)^X$$

V-natural in B. So the notion of cotensor generalises the notion of power rather than that of product. Up to equivalence, the V-category $V_{\aleph_1}^{op}$ is the free V-category with countable cotensors on 1.

Definition 41 *A countable Lawvere V-theory is a small V-category L with countable cotensors and a strong countable-cotensor preserving identity-on-objects V-functor $I : V_{\aleph_1}^{op} \longrightarrow L$.*

Theorem 42 *For any countable Lawvere V-theory L, the V-category L^{op} together with the V-functor $I^{op} : V_{\aleph_1} \longrightarrow L^{op}$ canonically support the structure of a Freyd-V-category.*

PROOF. It is shown in [6,7] that $V_{\aleph_1}$ has finite products and that $X \times Y$ is an X-tensor of Y, dualising the notion of cotensor. The V-category L^{op} has tensors, so one has a V-functor $X \otimes - : L^{op} \longrightarrow L^{op}$. Using duality, we are done.

Our analysis of $[\rightarrow, Set]$ in Sections 4 and 5 generalises routinely to $[\rightarrow, V]$: instead of speaking of the underlying ordinary category of an $[\rightarrow, Set]$-category, one speaks of the underlying V-category of an $[\rightarrow, V]$-category. However, enrichment of Section 6 requires more care: one must replace the conical colimits of Section 6 by V-weighted colimits, where V is regarded as a full sub-$[\rightarrow, V]$-category of $[\rightarrow, V]$. This can be confusing: V generalises Set and $[\rightarrow, V]$ generalises $[\rightarrow, Set]$, so V-weighted colimits generalise conical colimits and do not constitute all $[\rightarrow, V]$-weighted colimits.

A general analysis of weighted limits would be lengthy, so we refer the reader to the definitive book [6]. The upshot is given by the following results.

Theorem 43 *The free V-colimit completion of a small $[\to, V]$-category $J : C_0 \longrightarrow C_1$ is given by*

- *the V-category $[C_0^{op}, V]$*
- *the identity-on-objects/fully faithful factorisation of*

$$Lan_{J^{op}} : [C_0^{op}, V] \longrightarrow [C_1^{op}, V]$$

Theorem 44 *The free V-colimit completion of a small Freyd-V-category $J : C_0 \longrightarrow C_1$ is the free V-colimit-complete closed Freyd-V-category on J, i.e., the $[\to, V]$-category of Theorem 43 with a natural Freyd-V-structure.*

Finally, systematically replacing countable coproducts by countable tensors, we can enrich Section 7 to obtain the following decomposition result.

Theorem 45 *Let $J : V_{\aleph_1}^{op} \longrightarrow L$ be a countable Lawvere V-theory. Then the canonical closed Freyd-V-category*

$$V \longrightarrow Kl(T_L)$$

is the free V-colimit completion of $I^{op} : V_{\aleph_1} \longrightarrow L^{op}$ that preserves the countable tensors of I^{op}.

One can, of course, add recursion to the λ_c-calculus or to its first-order fragment and give a syntactic counterpart of our extension here from enrichment in *Set* to enrichment in ωCpo. Enrichment in ωCpo is orthogonal to the existence of solutions to recursive domain equations: the latter correspond to the existence of some colimits in a category or in an ωCpo-category. Not only do the closed V-categories we construct have such colimits, but also the V-category $V_{\aleph_1}$ used in the definition of Lawvere V-theory has them. The latter fact allows us to use such solutions as possible arities, as we do in Example 13.

References

[1] R. Blackwell, G.M. Kelly, and A.J. Power. Two-dimensional Monad Theory. *J. Pure Appl. Algebra*, 59:1–41, 1989.

[2] A. Carboni, S. Lack, and R. F. C. Walters. Introduction to Extensive and Distributive Categories. *J. Pure Appl. Algebra*, 84:145–158, 1993.

[3] M. Hyland, G. D. Plotkin, and A. J. Power. Combining Computational Effects: Commutativity and Sum. In *Proc. IFIP Conf. On Theoretical Computer Science*, pages 474–484, Kluwer, 2002.

[4] M. Hyland, G. D. Plotkin, and A. J. Power. Combining Computational Effects: Sum and Tensor. *Theoretical Computer Science*, to appear.

[5] Guen Bin Im and G.M. Kelly. A Universal Property of the Convolution Monoidal Structure. *J. Pure Appl. Algebra*, 43:75–88, 1986.

[6] G. M. Kelly. *Basic Concepts of Enriched Category Theory*, volume 64 of *London Mathematical Society Lecture Notes Series*, Cambridge University Press, 1982.

[7] G. M. Kelly. Structures Defined by Finite Limits in the Enriched Context. *Cahiers de Top. et Geom. Diff.*, 23: 3–42, 1982.

[8] Y. Kinoshita and A.J. Power. Data-refinement in Call-by-Value Languages. In *Proc. CSL*, in volume 1683 of *Lecture Notes in Computer Science*, pages 562–576, 1999.

[9] P. B. Levy, A. J. Power, and H. Thielecke. Modelling Environments in Call-by-Value Programming Languages. *Information and Computation*, 185:182–210, 2003.

[10] S. Mac Lane. *Categories for the Working Mathematician*, Berlin: Springer-Verlag, 1971.

[11] E. Moggi. Computational Lambda-Calculus and Monads. In *Proc. LICS 1989*, IEEE Computer Society Press, pages 14–23, 1989.

[12] E. Moggi. Notions of Computation and Monads. *Information and Computation*, 93:55–92, 1991.

[13] G.D. Plotkin and A.J. Power. Adequacy for Algebraic Effects. In *Proc. FOSSACS 2000*, in volume 2030 of *Lecture Notes in Computer Science*, pages 1–24, 2001.

[14] G.D. Plotkin and A.J. Power. Notions of Computation Determine Monads. In *Proc. FOSSACS 2001*, in volume 2303 of *Lecture Notes in Computer Science*, pages 342–356, 2002.

[15] G.D. Plotkin and A.J. Power. Semantics for Algebraic Operations. In *Proc. MFPS 2001*, in volume 45 of *Electronic Notes in Theoretical Computer Science*, 2001.

[16] G.D. Plotkin and A.J. Power. Algebraic Operations and Generic Effects. in *Applied Categorical Structures* 11:69–94, 2003.

[17] G. D. Plotkin and A. J. Power. Computational Effects and Operations: an Overview. in *Proc. Domains 2002*, in volume 73 of *Electronic Notes in Theoretical Computer Science*, 2002.

[18] G. D. Plotkin and A. J. Power. Logic for Computational Effects. in *Proc. IWFM 2003, British Computer Society Electronic Workshops in Computing*, 2003.

[19] A. J. Power. Enriched Lawvere Theories. *Theory and Applications of Categories*, 6:83–93, 1999.

[20] A.J. Power. Models for the Computational λ-Calculus. In *Proc. MFCSIT 2000*, in volume 40 of *Electronic Notes in Theoretical Computer Science*, 2001.

[21] A.J. Power. Premonoidal Categories as Categories with Algebraic Structure. *Theoretical Computer Science*, 278:303–321, 2002.

[22] A. J. Power. A Universal Embedding for the Higher Order Structure of Computational Effects. In *Proc. TLCA 2003*, in volume 2701 of *Lecture Notes in Computer Science*, pages 301–315, 2003.

[23] A. J. Power. Canonical Models for Computational Effects. In *Proc. FOSSACS 2004*, in volume 2987 of *Lecture Notes in Computer Science*, pages 438–452, 2004.

[24] A. J. Power. Countable Lawvere Theories and Computational Effects. Submitted.

[25] A. J. Power and E. P. Robinson. Premonoidal Categories and Notions of Computation. In *Proc. LDPL 96*, *Math Structures in Computer Science*, 7: 453–468, 1997.