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Centre for Intelligent Systems and their Applications

A Systematic Presentation of Quantified Modal Logics

by

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Informatics Research Report EDI-INF-RR-0150

School of Informatics **September 2002**

http://www.informatics.ed.ac.uk/

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Submitted to the Journal of the IGPL

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This paper provides a systematic presentation of Quantified Modal Logics (with constant domains and rigid designators). We therefore present a set of modular, uniform, normalising, sound and complete labelled sequent calculi for all QMLs whose frame properties can be expressed as a finite set of first-order sentences with equality.

We first present C-QK, a calculus for the logic QK, and then we extend it to any such logic QL. Each calculus, called C-QL, is modular (obtained by adding rules to C-QK), uniform (each added rule clearly relates to a property of the frame), normalising (frame reasoning only happens at the top of the proof tree) and Kripke-sound and complete for QL.

We improve on the existing literature on the subject (mainly, Vigano', "Labelled non-classical logics", 2000) by extending the class of logics for which such a presentation is given, and by giving a new proof of soundness and completeness.

Keywords : quantified modal logics, sequent calculi, labelled deductive systems, Kripke soundness, Kripke completeness, automated reasoning

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1 Introduction

Quantified Modal Logics (QMLs) are extensions of first-order classical logic in which one or more modal operators are introduced, and in which the notion of truth relies, in most of the existing literature, on the so-called *Kripke semantics*, or *semantics of the possible worlds*. This paper is an attempt at giving a systematic presentation of QMLs: uniform, intuitive, clear and complete for a class of QMLs as large as possible.

In order to achieve this goal, we have devised a family of labelled sequent calculi for QMLs (limited to constant domains and rigid designators, so far) which captures all logics whose frame properties can be expressed as a finite set of first-order sentences, with no restriction whatsoever on their shape, and possibly employing equality. Notwithstanding this generality, our sequent calculi retain some remarkable properties:

1. **modularity:** all calculi consist of a fixed base calculus for the weakest QML $\mathbf{QK}$, plus one sequent rule for each first-order sentence expressing a property of the frame. This, together with the use of labels, makes the presentation clear and intuitive. In case the property of the frame require equality, a few additional rules are added, without spoiling the modularity;

2. **uniformity:** each added rule is clearly related to the property it models, e.g., there is a rule for reflexivity, one for transitivity, etc. This avoids the need for rules obscurely enforcing frame properties, as is usually the case in unlabelled presentations;

3. **normalisation:** all calculi are normalising, in that the rules which model frame properties can be used just at the top of the proof tree without loss of completeness, therefore simplifying the presentation and potentially aiding automated deduction;

4. **soundness / completeness:** all calculi are proved sound and complete with respect to their frames; the proof of soundness and completeness is uniform, in that it is parametrised over the frame axioms.

1.1 Related work

It is indeed not the first time that a labelled presentation of a wide spectrum of QMLs is given; the most remarkable piece of work so far is due to Viganò ([Viganò, 2000]), who has given labelled Natural Deduction systems and sequent calculi for a wide set of QMLs. His systems are sound, complete and normalising for all QMLs whose frame properties can be axiomatised by first-order Horn clauses without equality (the so-called *relational theories*). We extend Viganò's work by giving sound, complete and normalising sequent calculi for all first-order axiomatisable QMLs, with no restriction on the shape of frame axioms and possibly employing equality between worlds; moreover, we employ a different way of proving soundness and completeness of such systems. It is worth remarking that Viganò's choice of restricting to relational theories is dictated exactly by the necessity of keeping a normalisation property to his systems (see his Theorem 4.3.7 and subsequent discussion in [Viganò, 2000]); our systems, on the other side, retain soundness, completeness and normalisation for a much wider set of QMLs. For example, no normalising system for $\mathbf{QS4.3}$ is given in [Viganò, 2000], whereas our system $\mathcal{C}_{\mathbf{QS4.3}}$ is sound and complete exactly for $\mathbf{QS4.3}$, and retains the normalisation property discussed in Subsection 3.4.

Labelled deduction was systematised by Gabbay ([Gabbay, 1996]) and has received quite a lot of attention since then, especially applied to non-classical logics like modal logics ([Viganò, 2000, Russo, 1996]), temporal logics ([Castellini and Smaill, 2001]), substructural logics ([Viganò, 2000]) and hybrid logics ([Blackburn, 2000]). The choice of labelled deduction is motivated by at least three reasons: *(i)* the explicit use of labels makes the presentation much more intuitive, in that it generates uniform sequent systems, *(ii)* it helps to keep reasoning on the properties of the frame separate from reasoning on logical formulae, thus potentially aiding automated deduction, *(iii)* in the quantified case, in which we are interested, it gives rise to systems which can be inherently more powerful than unlabelled ones: see for instance [Ghilardi, 1991], in which several unlabelled QMLs are proved incomplete with respect to their Kripke semantics. For this last reason, other remarkable work in unlabelled QMLs, (such as, e.g., [Fitting and Mendelsohn, 1998] and [Cerrito and Mayer, 2001]) is only loosely related with ours, although we have found it quite inspiring.

From now on we will indicate *Kripke*-soundness and completeness just by the terms *soundness* and *completeness*.

1.2 Outline of the paper

The paper is structured as follows: in Section 2 some preliminaries are given about the language of our logics, proof theory and QMLs; in Section 3 our sequent calculi are defined, and their benefits, *in primis* their normalisation property, are discussed; in Section 4 their soundness and completeness are stated and proved; lastly, in Section 5 we draw some conclusions and outline future work.

2 Preliminaries

In this Section we outline *(i)* the syntax of the language we will be using throughout the paper, *(ii)* the semantics of the logics generated by such language, *(iii)* a broad classification of QMLs, *(iv)* the basics of sequents and sequent calculi. Readers familiar with the area may skip this Section, which presents fairly standard concepts and terminology.

2.1 Syntax of the language

The syntax we present is standard in labelled deduction (see, e.g., [Gabbay, 1996]). Let $\mathcal{V}$, $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{V}_t$ be nonempty pairwise disjoint sets and let $\mathcal{F}_t$ be an empty set for now; then

Definition 1 (Formulae) *Logical terms* ($\mathbf{lt}$)*, logical atoms* ($\mathbf{la}$)*, logical formulae* ($\mathbf{lf}$)*, labels* ($\mathbf{lab}$)*, constraints* ($\mathbf{cst}$) *and* labelled formulae ($\mathbf{labf}$) *are defined according to the following grammar:*

$$
\begin{array}{rcll}
\mathbf{lt} & ::= & x \mid f(\mathbf{lt}_1,\ldots,\mathbf{lt}_n) & \textit{where } x\in\mathcal{V}, f\in\mathcal{F}, n\geq 0\\
\mathbf{la} & ::= & p(\mathbf{lt}_1,\ldots,\mathbf{lt}_m) & \textit{where } p\in\mathcal{P}, m\geq 0\\
\mathbf{lf} & ::= & \mathbf{la} \mid \neg\mathbf{lf} \mid \mathbf{lf}\supset\mathbf{lf} \mid \forall x.\mathbf{lf} \mid \Box\mathbf{lf} & \textit{where } x\in\mathcal{V}\\
\mathbf{lab} & ::= & 0 \mid t \mid g(\mathbf{lab}_1,\ldots,\mathbf{lab}_l) & \textit{where } t\in\mathcal{V}_t, 0, g\in\mathcal{F}_t, l\geq 0\\
\mathbf{cst} & ::= & \mathbf{lab}\prec\mathbf{lab} \mid \mathbf{lab}\doteq\mathbf{lab} & \\
\mathbf{labf} & ::= & \mathbf{lf} \mathbin{@} \mathbf{lab} &
\end{array}
$$

Labelled formulae and constraints are collectively called formulae *and denoted by* $\mathbf{forms}$.

Other connectives such as $\wedge$, $\vee$, $\leftrightarrow$, $\exists$ and $\Diamond$ are defined from the above ones in the usual way, e.g., $\exists$ is $\neg\forall\neg$, $\Diamond$ is $\neg\Box\neg$ and so on. Also, a standard notion of *free variables* of a formula is assumed, and formulae with no free variables are called *sentences*. Lastly, we will employ a standard notion of sub-formulae of a formula and of a set of formulae.

Examples of logical formulae are: $\forall x.\Diamond\exists y.r(x,y)$, $\Box p(a)$ and $\Box(p_1\wedge p_2)\leftrightarrow(\Box p_1\wedge\Box p_2)$, where $p,r,p_1,p_2\in\mathcal{P}$ and $a\in\mathcal{F}$; example of constraints are $\tau_1\prec\tau_2$ and $\tau\doteq\tau'$ where τ,τ',τ_1,τ_2 are labels; examples of labelled formulae are $p(a) \mathbin{@} 0$, $p_1\wedge p_2 \mathbin{@} \tau$ and $\forall x.p(x)\supset p(a) \mathbin{@} \tau'$. The $@$ operator is intended to bind less tightly than any other operator; the last example, for instance, means $\forall x.p(x)\supset p(a)$ holds at the world denoted by τ'.

2.2 Semantics and validity

We present a semantics which is largely based upon that given in [Abadí and Manna, 1990]. See also, e.g., Ghilardi and Corsi in [Bicchieri and Dalla Chiara, 1992].

Definition 2 (Structure) *We call a* structure *a tuple* $\mathbf{M}=\langle\mathcal{W},R,\mathcal{D},I\rangle$ *where:*

- $\mathcal{W}$ *is a nonempty set (the* set of possible worlds*);*
- $R\subseteq\mathcal{W}\times\mathcal{W}$ *(the* accessibility relation*);*
- $\mathcal{D}$ *is a nonempty set disjoint from* $\mathcal{W}$ *(the* domain of quantification*);*
- I *maps each world* $w\in\mathcal{W}$ *and predicate symbol* $p\in\mathcal{P}$ *to a predicate* $I(w,p)$ *over* $\mathcal{D}$*, and each function symbol* $f\in\mathcal{F}$ *to a* $\mathcal{D}$*-valued function* $I(f)$*.*

As is usual in modal logics, we will say that a structure has a property if and only if R in the structure has the property; for example, we will say that a structure is reflexive if and only if the associated R is reflexive, and so on. Note that, due to this semantics, the logics we consider have constant domains (i.e., the domain of quantification $\mathcal{D}$ is the same in all possible worlds) and rigid designators (i.e., the only "dynamic" objects are predicates).

Some more definitions: $\langle \mathcal{W}, R, \mathcal{D} \rangle$ is the *frame* on which the structure $\mathbf{M}$ is based. An *assignment* α is a function mapping variable symbols in $\mathcal{V}$ to values in $\mathcal{D}$. The assignment $\alpha^{[d/x]}$ assigns $d \in \mathcal{D}$ to x, leaving all the other symbols as in α. The *denotation* of a logical term s in the structure $\mathbf{M}$ w.r.t. α, written $s^{\mathbf{M},\alpha}$, is recursively defined as follows: if s is $v \in \mathcal{V}$, then $s^{\mathbf{M},\alpha} = \alpha(v)$; if s is $f(s_1, \dots, s_n)$, then $s^{\mathbf{M},\alpha} = I(f)(s_1^{\mathbf{M},\alpha}, \dots, s_n^{\mathbf{M},\alpha})$.

To give a semantics to labels and constraints, we introduce an interpretation I_l mapping $\prec$ to R, $\doteq$ to the equality relation, 0 to a distinguished element of $\mathcal{W}$ and function symbols in $\mathcal{F}_t$ to $\mathcal{W}$-valued functions, and an assignment α_l mapping variable symbols in $\mathcal{V}_t$ to elements of $\mathcal{W}$. The denotation of labels is analogous to that of logical terms (*de facto*, labels are terms of the labelling language): if τ is $t \in \mathcal{V}_t$, then $\tau^{I_l,\alpha_l} = \alpha_l(\tau)$; if τ is $g(\tau_1, \dots, \tau_n)$, then $\tau^{I_l,\alpha_l} = I_l(g)(\tau_1^{I_l,\alpha_l}, \dots, \tau_n^{I_l,\alpha_l})$. To ease the notation, we refer to elements of $\mathcal{W}$ with the letter w possibly decorated, and intend that $w, w_i, w', \dots$ are the objects denoted by labels $\tau, \tau_i, \tau', \dots$ That is, for example, $w = \tau^{I_l,\alpha_l}$.

Definition 3 (Truth in a structure) *A formula* φ *is* true in a structure $\mathbf{M}$ under the assignment α*, written* $\mathbf{M}, \alpha \models \varphi$*, if and only if:*

$$\begin{array}{lll}
\mathbf{M}, \alpha \models \tau_1 \prec \tau_2 & \textit{iff} & (w_1, w_2) \in R \\
\mathbf{M}, \alpha \models \tau_1 \doteq \tau_2 & \textit{iff} & w_1 = w_2 \\
\mathbf{M}, \alpha \models p(s_1, \dots, s_n) \mathbin{@} \tau & \textit{iff} & (s_1^{\mathbf{M},\alpha}, \dots, s_n^{\mathbf{M},\alpha}) \in I(w, p) \\
\mathbf{M}, \alpha \models \neg\varphi \mathbin{@} \tau & \textit{iff} & \textit{not} \;\; \mathbf{M}, \alpha \models \varphi \mathbin{@} \tau \\
\mathbf{M}, \alpha \models \varphi \supset \psi \mathbin{@} \tau & \textit{iff} & \textit{not} \;\; \mathbf{M}, \alpha \models \varphi \mathbin{@} \tau \;\; \textit{or} \;\; \mathbf{M}, \alpha \models \psi \mathbin{@} \tau \\
\mathbf{M}, \alpha \models \forall x.\varphi \mathbin{@} \tau & \textit{iff} & \textit{for all } d \in \mathcal{D}, \;\; \mathbf{M}, \alpha^{[d/x]} \models \varphi \mathbin{@} \tau \\
\mathbf{M}, \alpha \models \Box\varphi \mathbin{@} \tau & \textit{iff} & \textit{for all } w \in \mathcal{W}, \\
 & & \quad \textit{not} \;\; \mathbf{M}, \alpha \models \tau \prec t \;\; \textit{or} \;\; \mathbf{M}, \alpha \models \varphi \mathbin{@} t \\
 & & \quad\quad \textit{with } \alpha_l(t) = w
\end{array}$$

If a formula φ is true in $\mathbf{M}$ under all possible assignments α, we say that the structure $\mathbf{M}$ is a *model* for φ, and that φ is *true in the structure (model)* $\mathbf{M}$, written $\mathbf{M} \models \varphi$. Note that a sentence can only be true or false in a structure (model).

If a formula φ is true in all structures based on a frame $\mathbf{F}$, we say it is *valid on the frame* $\mathbf{F}$, written $\mathbf{F} \models \varphi$. Lastly, if it is valid on all frames belonging to a class of frames C, we say it is *valid on the class of frames* C and write $\models^C \varphi$. In particular, when a modal logic $\mathbf{QL}$ is known to correspond to a class of frames, we write $\models^{\mathbf{QL}} \varphi$. So, for instance, $\models^{\mathbf{QS4}} \varphi$ means that φ is valid on the class of transitive, reflexive frames, and so on.

2.3 Quantified Modal Logics

We will refer to Quantified Modal Logics with constant domains and rigid designators simply as QMLs or "logics" and will denote them as $\mathbf{QK}$, $\mathbf{QT}$ and so on. A thorough classification of their names, properties and characteristic axioms can be found, e.g., in [Cresswell and Hughes, 1995]. In the same book one can see that QMLs are usually organised in a hierarchy, in which $\mathbf{QK}$ is the weakest one (Table 1 in [Cresswell and Hughes, 1995]).

A relevant subset of them is characterised by classes of frames enjoying a set of properties which are expressible as a finite set of first-order sentences, possibly involving equivalence; Table 1 lists some of these properties, along with their corresponding names and first-order sentences[1]. Note that these sentences are naturally expressed in our language of labels.

[1] sentences and names of the properties are uniform with [Goldblatt, 1992], Chapter 1, except for the strong versions of weak density, directedness and connectedness, which are obtained by simply removing the antecedents of the implications, and atomicity, defined, e.g., in [van Benthem, 1984].

We will call such logics *FO-axiomatisable* and indicate them generically as $\mathbf{QL}$; we will say that the sentences which express their frame properties *axiomatise* them, and denote the set of those sentences as $\mathsf{FrmAx}(\mathbf{QL})$. If any of the sentences in $\mathsf{FrmAx}(\mathbf{QL})$ contains the symbol $\doteq$, we will say $\mathbf{QL}$ is a QML *with equality*, otherwise when necessary we will specify *without equality*.

Property (name)	Corresponding sentence
Seriality (D)	$\forall t \exists t'.t \prec t'$
Reflexivity (T)	$\forall t.t \prec t$
Irreflexivity	$\forall t.\neg\, t \prec t$
Symmetry (5)	$\forall t_0 t_1.t_0 \prec t_1 \supset t_1 \prec t_0$
Asymmetry	$\forall t_0 t_1.t_0 \prec t_1 \supset \neg\, t_1 \prec t_0$
Antisymmetry	$\forall t_0 t_1.(t_0 \prec t_1 \wedge t_1 \prec t_0) \supset t_0 \doteq t_1$
Transitivity (4)	$\forall t_0 t_1 t_2.(t_0 \prec t_1 \wedge t_1 \prec t_2) \supset t_0 \prec t_2$
Weak density	$\forall t_0 t_1.t_0 \prec t_1 \supset \exists t'.t_0 \prec t' \wedge t' \prec t_1$
Strong density	$\forall t_0 t_1 \exists t'.t_0 \prec t' \wedge t' \prec t_1$
Weak directedness (2)	$\forall t_0 t_1 t_2.(t_0 \prec t_1 \wedge t_0 \prec t_2) \supset \exists t'.t_1 \prec t' \wedge t_2 \prec t'$
Strong directedness	$\forall t_1 t_2 \exists t'.t_1 \prec t' \wedge t_2 \prec t'$
Weak connectedness (3)	$\forall t_0 t_1 t_2.(t_0 \prec t_1 \wedge t_0 \prec t_2) \supset (t_1 \prec t_2 \vee t_1 \doteq t_2 \vee t_2 \prec t_1)$
Strong connectedness	$\forall t_1 t_2.t_1 \prec t_2 \vee t_1 \doteq t_2 \vee t_2 \prec t_1$
Atomicity	$\forall t_1 \exists t'.t_1 \prec t' \wedge \forall t_2.t' \prec t_2 \supset t' \doteq t_2$

Table 1: properties of the accessibility relation as first-order sentences.

2.4 Sequent calculi and provability

We give now some basic definitions, uniform with [Troelstra and Schwichtenberg, 1996], Subsection 3.1. From now on, let Γ and Δ be finite multisets of formulae, with the convention that $\Gamma = \{\gamma_1, \ldots, \gamma_l\}, l \geq 0$ and $\Delta = \{\delta_1, \ldots, \delta_m\}, m \geq 0$.

A *sequent* is an expression $\Gamma \longrightarrow \Delta$. The γ_is are called *antecedents* and are intended conjunctively, while the δ_is are called *consequents* and are intended disjunctively; the sequent symbol can be read as a logical implication. Definition 3 and following are therefore straightforwardly extended to sequents: $\mathbf{M}, \alpha \models \Gamma \longrightarrow \Delta$ if and only if $\mathbf{M}, \alpha \models \gamma_1 \wedge \ldots \wedge \gamma_l \supset \delta_1 \vee \ldots \vee \delta_m$.

Let $n \geq 0$; then a *sequent rule* ρ is a pair (set of sequents, sequent), written

$$\frac{\Gamma_1 \longrightarrow \Delta_1 \quad \cdots \quad \Gamma_n \longrightarrow \Delta_n}{\Gamma \longrightarrow \Delta}\ \rho$$

where the $\Gamma_i \longrightarrow \Delta_i$'s are called *premises* and $\Gamma \longrightarrow \Delta$ is the *conclusion* of the rule. In displaying a sequent rule, generally, we highlight one formula in the conclusion (the *main* formula), and one or more formulae in each premise (the *active* formulae). The intuition is that the active formulae are introduced in the premises by manipulating the main formula via the sequent rule. We will use the term *frame rules* for rules whose active formulae are constraints, and *closing rules* for rules which have no premises. All other rules will be called *logical*.

A *sequent calculus* is a set of sequent rules. When displayed in a sequent calculus, a rule is really a *schema*, instantiated every time it appears in a derivation; one can think of the formulae and terms appearing in the rule as *placeholders*. We will follow the same convention, and implicitly use placeholders in sequent rules when displaying them in sequent calculi, throughout the paper. For a more formal treatment of this concept, see, e.g., [Kanger, 1983] or the seminal [Gentzen, 1969].

Assume a standard definition of *tree* (see, e.g., Subsection 2.2 of [Gallier, 1986]) and let $\mathcal{C}$ be a sequent calculus; then a *$\mathcal{C}$-derivation* of $\Gamma \longrightarrow \Delta$ is a tree in which every node N_i is labelled with a pair $\langle \rho_i, \Gamma_i \longrightarrow \Delta_i \rangle$, where $\rho_i \in \mathcal{C}$, and has n children, where n is the number of premises of ρ_i. The root node is labelled by $\Gamma \longrightarrow \Delta$ and the leaves have no labelling rule. Slightly abusing the language, we will say that N_i is labelled by ρ_i, by $\Gamma_i \longrightarrow \Delta_i$, or by a formula ϕ_i, if ϕ_i is main in ρ_i.

A *branch* of a derivation is a tuple of nodes $(N_1, \ldots, N_k)$ such that *(i)* N_1 is the root of the derivation, *(ii)* N_{i+1} is a child of N_i for all $i = 1, \ldots, N_{k-1}$ and *(iii)* N_k is a leaf of the derivation. A *closed* branch is a branch in which N_k is labelled by a closing rule. A *closed* $\mathcal{C}$-derivation of a sequent $\Gamma \longrightarrow \Delta$ (also called a $\mathcal{C}$-*proof* of $\Gamma \longrightarrow \Delta$) is a $\mathcal{C}$-derivation of $\Gamma \longrightarrow \Delta$ and whose branches are all closed.

Definition 4 (Provability) *If* $\Gamma \longrightarrow \Delta$ *has a* $\mathcal{C}$*-proof, we write*

$$\vdash_{\mathcal{C}} \Gamma \longrightarrow \Delta$$

and say that $\Gamma \longrightarrow \Delta$ *is* provable *in* $\mathcal{C}$ *(it is* $\mathcal{C}$*-provable), or that* $\Gamma \longrightarrow \Delta$ *is a* theorem *of* $\mathcal{C}$ *(it is a* $\mathcal{C}$*-theorem).*

Two proofs will be called *similar* if and only if they prove the same sequent. Finally, proof trees will be displayed, as is customary, bottom-up, that is, with the root at the bottom, labelled by the formula we want to prove.

3 Sequent calculi for QMLs

In this Section we introduce and develop $\mathcal{C}_{\mathbf{QK}}$, a sequent calculus for $\mathbf{QK}$; then a general procedure for strengthening $\mathcal{C}_{\mathbf{QK}}$ is outlined: first to QMLs without equality and then to all QMLs. A short discussion follows.

3.1 $\mathcal{C}_{\mathbf{QK}}$: a sequent calculus for QK

Assume from now on a standard definition of *substitution* of a variable in an expression E (formula, multiset of formulae, sequent) as presented in [Degtyarev and Voronkov, 2001], denoted $E[s/x]$ where s is a logical term or a label and x is, in turn, in $\mathcal{V}$ or in $\mathcal{V}_t$; then

Definition 5 ($\mathcal{C}_{\mathbf{QK}}$) *Let* $A \in \mathbf{forms}$, τ, τ_c *be labels,* φ, ψ *logical formulae and* c *a logical term; moreover, let* $a \in \mathcal{V}$ *and* $t_a \in \mathcal{V}_t$*; then* $\mathcal{C}_{\mathbf{QK}}$*, a sequent calculus for* $\mathbf{QK}$*, is visible in Table 2.*

$\mathcal{C}_{\mathbf{QK}}$ is a variant of Gentzen's sequent calculus LK for classical logic ([Gentzen, 1969]), except that

1. it is presented with no structural rules, but with a generalised closing rule and duplication of the main formula in $l\forall$ and $l\Box$ (in analogy, for instance, with system G in [Gallier, 1986], Definition 5.4.1);
2. it has two rules $r\Box$ and $l\Box$ for the $\Box$ operator, intuitively reflecting its semantics;
3. it is restricted to a minimal subset of operators ($\neg, \supset, \forall$ and $\Box$), with the assumption that rules for derived operators, such as $\exists$ and $\Diamond$, can be used here and there. They are obtained straightforwardly by composing rules in $\mathcal{C}_{\mathbf{QK}}$; for instance, $l\Diamond$ is obtained by considering the top and bottom sequents of the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \tau \prec t_a, \varphi \,@\, t_a \longrightarrow \Delta}{\Gamma, \tau \prec t_a \longrightarrow \neg\varphi \,@\, t_a, \Delta}\; r\neg}{\Gamma \longrightarrow \Box\neg\varphi \,@\, \tau, \Delta}\; r\Box}{\Gamma, \neg\Box\neg\varphi \,@\, \tau \longrightarrow \Delta}\; l\neg}{\Gamma, \Diamond\varphi \,@\, \tau \longrightarrow \Delta}\; \text{(definition of } \Diamond)$$

Closing rule

$$\frac{}{\Gamma, A \longrightarrow A, \Delta}\ \mathsf{ax}$$

Logical rules

$$\frac{\Gamma \longrightarrow \varphi \mathbin{@} \tau, \Delta}{\Gamma, \neg\varphi \mathbin{@} \tau \longrightarrow \Delta}\ l\neg \qquad \frac{\Gamma, \varphi \mathbin{@} \tau \longrightarrow \Delta}{\Gamma \longrightarrow \neg\varphi \mathbin{@} \tau, \Delta}\ r\neg$$

$$\frac{\Gamma, \psi \mathbin{@} \tau \longrightarrow \Delta \quad \Gamma \longrightarrow \varphi \mathbin{@} \tau, \Delta}{\Gamma, \varphi \supset \psi \mathbin{@} \tau \longrightarrow \Delta}\ l\supset \qquad \frac{\Gamma, \varphi \mathbin{@} \tau \longrightarrow \psi \mathbin{@} \tau, \Delta}{\Gamma \longrightarrow \varphi \supset \psi \mathbin{@} \tau, \Delta}\ r\supset$$

$$\frac{\Gamma, \forall x.\varphi \mathbin{@} \tau, \varphi[c/x] \mathbin{@} \tau \longrightarrow \Delta}{\Gamma, \forall x.\varphi \mathbin{@} \tau \longrightarrow \Delta}\ l\forall \qquad \frac{\Gamma \longrightarrow \varphi[a/x] \mathbin{@} \tau, \Delta}{\Gamma \longrightarrow \forall x.\varphi \mathbin{@} \tau, \Delta}\ r\forall$$

$$\frac{\Gamma, \Box\varphi \mathbin{@} \tau, \varphi \mathbin{@} \tau_c \longrightarrow \Delta \quad \Gamma, \Box\varphi \mathbin{@} \tau \longrightarrow \tau \prec \tau_c, \Delta}{\Gamma, \Box\varphi \mathbin{@} \tau \longrightarrow \Delta}\ l\Box \qquad \frac{\Gamma, \tau \prec t_a \longrightarrow \varphi \mathbin{@} t_a, \Delta}{\Gamma \longrightarrow \Box\varphi \mathbin{@} \tau, \Delta}\ r\Box$$

Table 2: the calculus $\mathcal{C}_{\mathbf{QK}}$ for $\mathbf{QK}$. $A \in \mathbf{forms}$, τ, τ_c are labels, φ, ψ logical formulae and c a logical term. $a \in \mathcal{V}$ and $t_a \in \mathcal{V}_t$ cannot appear free in the conclusion of $r\forall$ and $r\Box$.

It is possible to prove in $\mathcal{C}_{\mathbf{QK}}$ a number of characteristic axioms of $\mathbf{QK}$; as an example, in Figure 1 we sketch he $\mathcal{C}_{\mathbf{QK}}$-proof of *Modal Modus Ponens*, and in Figure 2 the $\mathcal{C}_{\mathbf{QK}}$-proof of the *Converse Barcan Formula*, characteristic of quantified modal logics with constant domains[2]. Also, the *Rule of Necessitation* is naturally enforced: for any logical formula φ and label τ, if $\vdash_{\mathcal{C}_{\mathbf{QK}}} \varphi \mathbin{@} \tau$ then $\vdash_{\mathcal{C}_{\mathbf{QK}}} \Box\varphi \mathbin{@} \tau$. This can be easily shown by structural induction.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\psi \mathbin{@} t_a \longrightarrow \psi \mathbin{@} t_a}\ \mathsf{ax} \quad \dfrac{}{\varphi \mathbin{@} t_a \longrightarrow \varphi \mathbin{@} t_a}\ \mathsf{ax}}{\varphi \mathbin{@} t_a, \varphi \supset \psi \mathbin{@} t_a \longrightarrow \psi \mathbin{@} t_a}\ l\supset \quad \dfrac{}{0 \prec t_a \longrightarrow 0 \prec t_a}\ \mathsf{ax}}{0 \prec t_a, \varphi \mathbin{@} t_a, \Box(\varphi \supset \psi) \mathbin{@} 0 \longrightarrow \psi \mathbin{@} t_a}\ l\Box \quad \dfrac{}{0 \prec t_a \longrightarrow 0 \prec t_a}\ \mathsf{ax}}{0 \prec t_a, \Box\varphi \mathbin{@} 0, \Box(\varphi \supset \psi) \mathbin{@} 0 \longrightarrow \psi \mathbin{@} t_a}\ l\Box}{\Box\varphi \mathbin{@} 0, \Box(\varphi \supset \psi) \mathbin{@} 0 \longrightarrow \Box\psi \mathbin{@} 0}\ r\Box}{\Box\varphi \wedge \Box(\varphi \supset \psi) \mathbin{@} 0 \longrightarrow \Box\psi \mathbin{@} 0}\ l\wedge}{\longrightarrow \Box\varphi \wedge \Box(\varphi \supset \psi) \supset \Box\psi \mathbin{@} 0}\ r\supset}{}}{}$$

Figure 1: a $\mathcal{C}_{\mathbf{QK}}$-proof of Modal Modus Ponens.

3.2 Sequent calculi for QMLs without equality

Assume standard notions of *prenex normal form* and *Skolemisation* of a first-order formula (see, e.g., [Shoenfield, 1970]); then we introduce the following procedure which builds a sequent rule out of a first-order sentence:

Definition 6 (Strengthening) *Let ϕ be a first-order sentence in the language of labels not containing the equality symbol; then the* strengthening procedure*, yielding sequent rule* $\mathsf{Str}(\phi)$*, is defined as follows:*

[2] all proof sketches in the paper, although some unessential formulae may be omitted here and there for the sake of conciseness, are completely rigorous. Especially, we will leave out the copy of the main formula in the premises of rules $l\forall$ and $l\Box$, when they are not necessary.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(a) \mathbin{@} t_a \longrightarrow p(a)@t_a}\,\mathsf{ax}}{\forall x.p(x) \mathbin{@} t_a \longrightarrow p(a)@t_a}\,l\forall \quad \dfrac{}{0 \prec t_a \longrightarrow 0 \prec t_a}\,\mathsf{ax}}{0 \prec t_a, \Box\forall x.p(x) \mathbin{@} 0 \longrightarrow p(a)@t_a}\,l\Box}{\Box\forall x.p(x) \mathbin{@} 0 \longrightarrow \Box p(a)@0}\,r\Box}{\Box\forall x.p(x) \mathbin{@} 0 \longrightarrow \forall x.\Box p(x)@0}\,r\forall}{\longrightarrow \Box\forall x.p(x) \supset \forall x.\Box p(x)@0}\,r\supset$$

Figure 2: a $\mathcal{C}_{\mathbf{QK}}$-proof of the Converse Barcan Formula.

1. *convert ϕ into prenex normal form and skolemise; call the new sentence ϕ^S;*
2. *add to $\mathcal{F}_t$ the Skolem function(s) introduced at the previous step;*
3. *build a $2LK$-derivation of $\Gamma, \phi^S \longrightarrow \Delta$ (see Table 5 in Subsection 4.1) in which every sequent labelling a leaf contains only constraints, Γ or Δ; when using rule $l\forall^*_\theta$, avoid copying the main formula into the premise;*
4. *finally, let $\Gamma \longrightarrow \Delta$ be the conclusion of $\mathsf{Str}(\phi)$, and let the leaves of the derivation obtained at the previous step be its premises.*

Note that rules obtained by the strengthening procedure are frame rules. This is appropriate, since they express properties of the accessibility relation and, in turn, of the frame. As already noted, when displayed in sequent calculi, rules are schemas; we will denote the placeholders by the letter τ, possibly decorated, meaning any label.

As an example of the strengthening procedure, consider a sentence not involving equality in Table 1, for instance 2; in order to obtain $\mathsf{Str}(2)$, first build its prenex normal form and skolemise:

$$2^S = \forall t_0 t_1 t_2.(t_0 \prec t_1 \wedge t_0 \prec t_2) \supset t_1 \prec cv(t_0, t_1, t_2) \wedge t_2 \prec cv(t_0, t_1, t_2)$$

Let then $\mathcal{F}_t = \{cv\}$; now "unfold" 2^S as shown in Figure 3; finally, build $\mathsf{Str}(2)$ by taking $\Gamma \longrightarrow \Delta$ as the conclusion, and the leaves of the proof tree in Figure 3 as the premises.

$$\dfrac{\dfrac{\dfrac{\Gamma \longrightarrow \tau_0 \prec \tau_1, \Delta \quad \Gamma \longrightarrow \tau_0 \prec \tau_2, \Delta}{\Gamma, \longrightarrow \tau_0 \prec \tau_1 \wedge \tau_0 \prec \tau_2, \Delta}\,r\wedge^* \quad \dfrac{\Gamma, \tau_1 \prec cv(\tau_0, \tau_1, \tau_2), \tau_2 \prec cv(\tau_0, \tau_1, \tau_2) \longrightarrow \Delta}{\Gamma, \tau_1 \prec cv(\tau_0, \tau_1, \tau_2) \wedge \tau_2 \prec cv(\tau_0, \tau_1, \tau_2) \longrightarrow \Delta}\,l\wedge^*}{\Gamma, (\tau_0 \prec \tau_1 \wedge \tau_0 \prec \tau_2) \supset \tau_1 \prec cv(\tau_0, \tau_1, \tau_2) \wedge \tau_2 \prec cv(\tau_0, \tau_1, \tau_2) \longrightarrow \Delta}\,l\supset^*}{\Gamma, \forall t_0 t_1 t_2.(t_0 \prec t_1 \wedge t_0 \prec t_2) \supset t_1 \prec cv(t_0, t_1, t_2) \wedge t_2 \prec cv(t_0, t_1, t_2) \longrightarrow \Delta}\,l\forall^*_\theta l\forall^*_\theta l\forall^*_\theta$$

Figure 3: application of Step 3 of the strengthening procedure to the sentence 2^S. The leaves of this derivation are the premises of rule $\mathsf{Str}(2)$, called wdir and visible in Table 4.

Note that there is no restriction on the shape of ϕ, except that it must not contain the equality symbol so far. An interesting issue is that

Proposition 7 (Termination of strengthening) *The strengthening procedure is terminating.*

Proof: it suffices to show that every step of the procedure is terminating. Obviously, the only non-trivially terminating step is Step 3: for this, note that every application of a $2LK$ rule to ϕ^S reduces the number of connectives in it (provided that copying the main formula in $l\forall^*_\theta$ is forbidden), eventually leading to a set of sequents which contain only constraints. •

Frame rules

$$\frac{}{\Gamma \longrightarrow \tau \doteq \tau, \Delta}\ \mathsf{refl}_{\doteq}$$

$$\frac{\Gamma[\tau'/t], \tau \doteq \tau' \longrightarrow \Delta[\tau'/t]}{\Gamma[\tau/t], \tau \doteq \tau' \longrightarrow \Delta[\tau/t]}\ \mathsf{sub}_{\doteq}$$

Table 3: rules for equality. $\mathcal{C}^{\doteq}_{\mathbf{QK}}$ is the union of these rules and $\mathcal{C}_{\mathbf{QK}}$. τ, τ' are labels and $t \in \mathcal{V}_t$. In rule $\mathsf{sub}_{\doteq}$, the occurrences of τ replaced by τ' are in logical atoms or constraints only.

The restriction on the use of duplicate formulae in rule $l\forall^*_\theta$ at Step 3 of the procedure is now clear; this apparent weakening is taken into account later on in the proof of completeness (see Section 4).

For any QML without equality $\mathbf{QL}$, let $\mathsf{Str}(\mathbf{QL})$ be the sequent calculus obtained by strengthening the sentences in $\mathsf{FrmAx}(\mathbf{QL})$, that is $\mathsf{Str}(\mathbf{QL}) = \{\rho \mid \rho = \mathsf{Str}(\phi), \phi \in \mathsf{FrmAx}(\mathbf{QL})\}$. Then a sequent calculus for $\mathbf{QL}$ can be built by taking $\mathcal{C}_{\mathbf{QK}} \cup \mathsf{Str}(\mathbf{QL})$.

This calculus is *modular*, in that it is obtained by adding to the (unchanged) basic calculus $\mathcal{C}_{\mathbf{QK}}$ a set of new rules, and *uniform*, in that (as Definition 6 suggests) each sequent rule in $\mathsf{Str}(\mathbf{QL})$ is clearly and intuitively related to a first-order sentence enforcing a property of the frame.

Moreover, by Proposition 7 and the fact that $\mathsf{FrmAx}(\mathbf{QL})$ is finite, we have that calculi obtained this way are *finitary*, in that they have a finite number of rules, and each rule has a finite number of premises.

As a simple example, in Figure 4 we sketch the proof of axiom $\Box p \supset \Box\Box p \text{ @ } 0$, characteristic of transitive frames, in $\mathcal{C}_{\mathbf{QK}} \cup \{\mathsf{trans}\}$. Rule $\mathsf{trans} = \mathsf{Str}(4)$ is visible in Table 4.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{p@t_2 \longrightarrow p@t_2}\ \mathsf{ax} \qquad \cfrac{\cfrac{}{0 \prec t_1 \longrightarrow 0 \prec t_1}\ \mathsf{ax} \quad \cfrac{}{t_1 \prec t_2 \longrightarrow t_1 \prec t_2}\ \mathsf{ax} \quad \cfrac{}{0 \prec t_2 \longrightarrow 0 \prec t_2}\ \mathsf{ax}}{0 \prec t_1, t_1 \prec t_2 \longrightarrow 0 \prec t_2}\ \mathsf{trans}}{0 \prec t_1, t_1 \prec t_2, \Box p@0 \longrightarrow p@t_2}\ l\Box}{0 \prec t_1, \Box p@0 \longrightarrow \Box p@t_1}\ r\Box}{\Box p@0 \longrightarrow \Box\Box p@0}\ r\Box}{\longrightarrow \Box p \supset \Box\Box p@0}\ r\supset$$

Figure 4: a proof of axiom $\Box p \supset \Box\Box p \text{ @ } 0$, characteristic of transitive frames, in $\mathcal{C}_{\mathbf{QK}} \cup \{\mathsf{trans}\}$.

3.3 Sequent calculi for QMLs with equality

The equality symbol between labels is already present in our syntax (Definition 1) and it has a semantics (Definition 3). Now

Definition 8 ($\mathcal{C}^{\doteq}_{\mathbf{QK}}$) *Let τ, τ' be labels and $t \in \mathcal{V}_t$; then $\mathcal{C}^{\doteq}_{\mathbf{QK}}$, a sequent calculus for $\mathbf{QK}$ augmented for equality between labels, is the union of $\mathcal{C}_{\mathbf{QK}}$ (recall Table 2) and the rules visible in Table 3.*

Rules in Table 3 enforce basic properties of $\doteq$, for instance that assuming $\tau \doteq \tau'$, a label τ can be uniformly substituted with τ'[3]. Note that rule $\mathsf{sub}_{\doteq}$ is included in the set of frame rules altough it can have active logical atoms; we choose to do this because both $\mathsf{refl}_{\doteq}$ and $\mathsf{sub}_{\doteq}$ deal with the symbol of equality, which is defined only between labels.

Definition 6 carries on straightforwardly for all QMLs (just remove the words "not containing the equality symbol" from it). Same goes for Proposition 7. For any QML $\mathbf{QL}$, now possibly with equality, a

[3]the restriction to logical atoms and constraints is dictated by the completeness argument and will be clarified in Section 4.

sequent calculus for $\mathbf{QL}$ can be built by taking $\mathcal{C}^{\doteq}_{\mathbf{QK}} \cup \mathsf{Str}(\mathbf{QL})$. All properties defined and proved in the previous Subsection still hold: all calculi obtained as described above are modular, uniform and finitary.

As a non-completely trivial example, in Figure 5 we sketch the proof of axiom $\Diamond\Box p \supset \Box\Diamond p \ @\ 0$, characteristic of the logic of reflexive, weakly directed frames, in $\mathcal{C}^{\doteq}_{\mathbf{QK}} \cup \{\mathsf{refl}, \mathsf{wconn}\}$. Rules $\mathsf{refl} = \mathsf{Str}(T)$ and $\mathsf{wconn} = \mathsf{Str}(3)$ are visible in Table 4. This proof is possible, as we expect, since the property of weak connectedness is strictly stronger than that of weak directedness. Note the use of $\doteq$.

$$
\dfrac{\dfrac{\dfrac{}{p \ @\ t_1 \longrightarrow p \ @\ t_1}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{t_1 \prec t_1 \longrightarrow t_1 \prec t_1}\,\mathsf{ax}}{\longrightarrow t_1 \prec t_1}\,\mathsf{refl}}{\Box p \ @\ t_1 \longrightarrow p \ @\ t_1}\,l\Box \qquad \dfrac{\dfrac{}{t_1 \prec t_1 \longrightarrow t_1 \prec t_1}\,\mathsf{ax}}{\longrightarrow t_1 \prec t_1}\,\mathsf{refl}}{\dfrac{\Box p \ @\ t_1 \longrightarrow \Diamond p \ @\ t_1}{t_1 \doteq t_2, \Box p \ @\ t_1 \longrightarrow \Diamond p \ @\ t_2}\,\mathsf{sub}_{\doteq}}\,r\Diamond
$$

Branch $\boxed{3}$

$$
\dfrac{\dfrac{\dfrac{}{p \ @\ t_1 \longrightarrow p \ @\ t_1}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{t_1 \prec t_1 \longrightarrow t_1 \prec t_1}\,\mathsf{ax}}{\longrightarrow t_1 \prec t_1}\,\mathsf{refl}}{t_2 \prec t_1, \Box p \ @\ t_1 \longrightarrow p \ @\ t_1}\,l\Box \qquad \dfrac{}{t_2 \prec t_1 \longrightarrow t_2 \prec t_1}\,\mathsf{ax}}{t_2 \prec t_1, \Box p \ @\ t_1 \longrightarrow \Diamond p \ @\ t_2}\,r\Diamond
$$

Branch $\boxed{2}$

$$
\dfrac{\dfrac{\dfrac{}{p \ @\ t_2 \longrightarrow p \ @\ t_2}\,\mathsf{ax} \qquad \dfrac{}{t_1 \prec t_2 \longrightarrow t_1 \prec t_2}\,\mathsf{ax}}{t_1 \prec t_2, \Box p \ @\ t_1 \longrightarrow p \ @\ t_2}\,l\Box \qquad \dfrac{\dfrac{}{t_2 \prec t_2 \longrightarrow t_2 \prec t_2}\,\mathsf{ax}}{t_2 \prec t_2}\,\mathsf{refl}}{t_1 \prec t_2, \Box p \ @\ t_1 \longrightarrow \Diamond p \ @\ t_2}\,r\Diamond
$$

Branch $\boxed{1}$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{0 \prec t_1 \longrightarrow 0 \prec t_1}\,\mathsf{ax} \quad \dfrac{}{0 \prec t_2 \longrightarrow 0 \prec t_2}\,\mathsf{ax} \quad \boxed{1} \quad \boxed{2} \quad \boxed{3}}{0 \prec t_1, 0 \prec t_2, \Box p \ @\ t_1 \longrightarrow \Diamond p \ @\ t_2}\,\mathsf{wconn}}{0 \prec t_1, \Box p \ @\ t_1 \longrightarrow \Box\Diamond p \ @\ 0}\,r\Box}{\Diamond\Box p \ @\ 0 \longrightarrow \Box\Diamond p \ @\ 0}\,l\Diamond}{\longrightarrow \Diamond\Box p \supset \Box\Diamond p \ @\ 0}\,r\supset
$$

Bottom of the tree

Figure 5: a proof of axiom $\Diamond\Box p \supset \Box\Diamond p \ @\ 0$, characteristic of reflexive, weakly directed frames, in $\mathcal{C}^{\doteq}_{\mathbf{QK}} \cup \{\mathsf{refl}, \mathsf{wconn}\}$. The bottom subtree is at the root of the proof; the three subtrees above correspond to placeholders $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$.

3.4 The entailment rule: normalisation

Lastly, we introduce a rule which takes into account all frame rules seen so far. Let $\mathsf{FrmRl}(\mathbf{QL})$ be the union of $\mathsf{Str}(\mathbf{QL})$ and the rules in Table 3; then

Definition 9 ($\mathcal{C}_{\mathbf{QL}}$ and the entailment rule) *For each logic* $\mathbf{QL}$, *let* $\mathcal{C}_{\mathbf{QL}} = \mathcal{C}_{\mathbf{QK}} \cup \mathsf{ent}_{\mathbf{QL}}$, *where* $\mathsf{ent}_{\mathbf{QL}}$ *(*entailment*) is the following rule:*

$$\frac{}{\Gamma \longrightarrow \Delta}\ \mathsf{ent}_{\mathbf{QL}} \quad \textit{(side condition: } \vdash_{\mathsf{FrmRl}(\mathbf{QL})} \Gamma \longrightarrow \Delta\textit{)}$$

According to the above Definition, in each $\mathcal{C}_{\mathbf{QL}}$-proof, rule $\mathsf{ent}_{\mathbf{QL}}$ represents a subproof in which only rules in $\mathsf{FrmRl}(\mathbf{QL})$ are used. In $\mathcal{C}_{\mathbf{QL}}$ there is a strong restriction on the use of frame rules, which, since $\mathsf{ent}_{\mathbf{QL}}$ is a closing rule, cannot be followed by the application of any logical rule. In other words,

Proposition 10 (Normalisation) *For each FO-axiomatisable logic* $\mathbf{QL}$*,* $\mathcal{C}_{\mathbf{QL}}$*-proofs are* normal*, in the sense that no logical rules are ever used above a frame rule.*

Proof: trivial, from the facts that *(i)* no frame rules appear in any $\mathcal{C}_{\mathbf{QL}}$, and that *(ii)* $\mathsf{ent}_{\mathbf{QL}}$ is a closing rule. •

Again, all calculi $\mathcal{C}_{\mathbf{QL}}$ retain the properties defined and proved in the previous Subsections: they are still modular, uniform and finitary.

3.5 Discussion

The methodology outlined earlier on allows us to build sequent calculi for any FO-axiomatisable QML (with or without equality). As an extended example, Table 4 shows the rules obtained by application of the strengthening procedure to sentences in Table 1. We have given them mnemonic names, such as $\mathsf{refl} = \mathsf{Str}(T)$, and so on. As usual, labels in the rules of Table 4 are really placeholders.

Besides adding to the elegance of the presentation, the properties of modularity and uniformity are useful for the implementation of these logics. Such an implementation would indeed benefit from not having to be redone from scratch each time a new, stronger logic is needed; modularity could be reflected in modularity of the automated machinery.

Moreover, the property of normalisation reduces the search space during proof search in any $\mathcal{C}_{\mathbf{QL}}$. In principle, rule $\mathsf{ent}_{\mathbf{QL}}$ can be replaced by any reasoning method whatsoever for the first-order theory of $\mathsf{FrmAx}(\mathbf{QL})$, seen as a black box; in particular, any efficient machinery for equivalence reasoning can be employed. Normal proofs here can be seen as a generalised version of *regular* proofs in sequent calculi for logics with equality, an issue addressed, e.g., in [Degtyarev and Voronkov, 2001].

Although not in the scope of this paper, this is an important issue, whenever we plan to do automated reasoning using these sequent calculi; of course, to do any automated reasoning whatsoever in this framework, one needs more techniques developed in the Automated Reasoning community than what is described in this paper, for example the use of metavariables or free variables (see, e.g., [Shankar, 1992, Degtyarev and Voronkov, 2001]). In fact $\mathcal{C}_{\mathbf{QS4.3}}$ is currently being used by the authors as the starting point for an experiment in automated deduction for formal methods (see [Castellini and Smaill, 2002]).

Lastly, we show two more non-trivial examples. First, we recast the (non-normal) proof in Figure 5 in $\mathcal{C}_{\mathbf{QS4.3}}$; the result is visible in Figure 6, where, still, we indicate explicitly the use of frame rules, instead of using rule $\mathsf{ent}_{\mathbf{QS4.3}}$, for the sake of clarity; note however that, as expected, no logical rules are used above any frame rule, i.e., this proof *is* normal.

Second, in Figure 7 we show that McKinsey's axiom, $\Box\Diamond p \supset \Diamond\Box p$, is provable in $\mathcal{C}_{\mathbf{QS4.1}}$, a calculus for logic $\mathbf{QS4.1}$ in which $\mathsf{FrmRl}(\mathbf{QS4.1}) = \{\mathsf{refl}_{\doteq}, \mathsf{sub}_{\doteq}, \mathsf{refl}, \mathsf{trans}, \mathsf{atom}\}$ (see, e.g., [van Benthem, 1984]). As we expect, this proof is also normal.

4 Soundness and completeness

Recall Definitions 3 and 4, and let $\mathbf{QL}$ be any FO-axiomatisable QML (with or without equality); in this Section we prove that $\mathcal{C}_{\mathbf{QL}}$ is sound and complete for each $\mathbf{QL}$, that is, whatever is $\mathcal{C}_{\mathbf{QL}}$-provable is $\mathbf{QL}$-valid and vice-versa:

$$\vdash_{\mathcal{C}_{\mathbf{QL}}} \Gamma \longrightarrow \Delta \quad \text{iff} \quad \models^{\mathbf{QL}} \Gamma \longrightarrow \Delta.$$

$$\frac{\Gamma, \tau \prec wit(\tau) \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}\ \textsf{ser}$$

$$\frac{\Gamma, \tau \prec \tau \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}\ \textsf{refl}$$

$$\frac{\Gamma \longrightarrow \tau \prec \tau, \Delta}{\Gamma \longrightarrow \Delta}\ \textsf{irr}$$

$$\frac{\Gamma \longrightarrow \tau_0 \prec \tau_1, \Delta \quad \Gamma, \tau_1 \prec \tau_0 \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}\ \textsf{symm}$$

$$\frac{\Gamma \longrightarrow \tau_0 \prec \tau_1, \Delta \quad \Gamma \longrightarrow \tau_1 \prec \tau_0, \Delta}{\Gamma \longrightarrow \Delta}\ \textsf{asymm}$$

$$\frac{\Gamma \longrightarrow \tau_0 \prec \tau_1, \Delta \quad \Gamma \longrightarrow \tau_1 \prec \tau_0, \Delta \quad \Gamma, \tau_0 \doteq \tau_1 \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}\ \textsf{antisymm}$$

$$\frac{\Gamma \longrightarrow \tau_0 \prec \tau_1, \Delta \quad \Gamma \longrightarrow \tau_1 \prec \tau_2, \Delta \quad \Gamma, \tau_0 \prec \tau_2 \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}\ \textsf{trans}$$

$$\frac{\Gamma \longrightarrow \tau_0 \prec \tau_1, \Delta \quad \Gamma, \tau_0 \prec hb(\tau_0, \tau_1), hb(\tau_0, \tau_1) \prec \tau_1 \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}\ \textsf{wdens}$$

$$\frac{\Gamma, \tau_0 \prec hb(\tau_0, \tau_1), hb(\tau_0, \tau_1) \prec \tau_1 \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}\ \textsf{sdens}$$

$$\frac{\Gamma \longrightarrow \tau_0 \prec \tau_1, \Delta \quad \Gamma \longrightarrow \tau_0 \prec \tau_2, \Delta \quad \Gamma, \tau_1 \prec cv(\tau_0, \tau_1, \tau_2), \tau_2 \prec cv(\tau_0, \tau_1, \tau_2) \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}\ \textsf{wdir}$$

$$\frac{\Gamma, \tau_1 \prec cv(\tau_0, \tau_1, \tau_2), \tau_2 \prec cv(\tau_0, \tau_1, \tau_2) \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}\ \textsf{sdir}$$

$$\frac{\Gamma \longrightarrow \tau_0 \prec \tau_1, \Delta \quad \Gamma \longrightarrow \tau_0 \prec \tau_2, \Delta \quad \Gamma, \tau_1 \prec \tau_2 \longrightarrow \Delta \quad \Gamma, \tau_1 \doteq \tau_2 \longrightarrow \Delta \quad \Gamma, \tau_2 \prec \tau_1 \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}\ \textsf{wconn}$$

$$\frac{\Gamma, \tau_1 \prec \tau_2 \longrightarrow \Delta \quad \Gamma, \tau_1 \doteq \tau_2 \longrightarrow \Delta \quad \Gamma, \tau_2 \prec \tau_1 \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}\ \textsf{sconn}$$

$$\frac{\Gamma, \tau_1 \prec la(\tau_1), la(\tau_1) \doteq \tau_2 \longrightarrow \Delta \quad \Gamma, \tau_1 \prec la(\tau_1) \longrightarrow la(\tau_1) \prec \tau_2, \Delta}{\Gamma \longrightarrow \Delta}\ \textsf{atom}$$

Table 4: frame rules obtained from sentences in Table 1 via the strengthening procedure. wit (the "witness" world), hb (the world "halfway between"), cv (the "convergent" world) and la (the "last" world) are Skolem functions, purposefully added to $\mathcal{F}_t$ by the strengthening procedure.

4.1 Two-sorted first-order logic with equality

We now sketch two-sorted first-order logic with equality on one sort (which we call $2FOL$) and an associated sequent calculus. This machinery is needed for the proof of soundness and completeness. The following presentation is rather informal; the reader can check the details in [Gallier, 1986] and [Degtyarev and Voronkov, 2001], which are the main sources of inspiration.

The vocabulary of $2FOL$ has three sets $\mathcal{V}'$, $\mathcal{F}'$ and $\mathcal{P}'$ of variable, function and predicate symbols, plus two symbols, ι and θ, called *sort symbols*; to each function and predicate symbol is associated an n-uple in $\{\iota, \theta\}^n$ (the *rank* of the symbol — see [Gallier, 1986], Subsection 5.2.1). Informally: the rank of a symbol associates a sort (or *type*) of each argument of the function or predicate associated with the symbol; for function symbols, it also states the type of the function itself. By default, $=_\iota \in \mathcal{P}'$ with rank (ι, ι). $=_\iota$ denotes equivalence among elements of sort ι.

The language of $2FOL$ is built out of terms and atoms into formulae by means of $\neg, \supset$ and $\forall$, analogously to what happens in first-order logic, but respecting the rank of each symbol.

A *structure* of $2FOL$ is a pair $\mathbf{M}' = \langle \mathcal{D}', I' \rangle$ in which $\mathcal{D}' = \mathcal{W}' \cup \mathcal{C}'$ where $\mathcal{W}'$ and $\mathcal{C}'$ are disjoint and are called *sorts*. Every term of $2FOL$ is associated via its rank to exactly one sort; we indicate this fact

$$\dfrac{\dfrac{\dfrac{}{t_1 \prec t_1 \longrightarrow t_1 \prec t_1}\;\mathsf{ax}}{\longrightarrow t_1 \prec t_1}\;\mathsf{refl}}{t_1 \doteq t_2 \longrightarrow t_1 \prec t_2}\;\mathsf{sub}_{\doteq}$$

Branch $\boxed{2}$

$$\dfrac{\dfrac{}{0 \prec t_1 \longrightarrow 0 \prec t_1}\;\mathsf{ax} \quad \dfrac{}{0 \prec t_2 \longrightarrow 0 \prec t_2}\;\mathsf{ax} \quad \dfrac{}{t_1 \prec t_2 \longrightarrow t_1 \prec t_2}\;\mathsf{ax} \quad \boxed{2} \quad \dfrac{}{t_2 \prec t_1 \longrightarrow t_2 \prec t_1}\;\mathsf{ax}}{0 \prec t_1, 0 \prec t_2 \longrightarrow t_2 \prec t_1, t_1 \prec t_2}\;\mathsf{wconn}$$

Branch $\boxed{1}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \,@\, t_2 \longrightarrow p \,@\, t_2}\;\mathsf{ax} \quad \dfrac{\dfrac{}{t_2 \prec t_2 \longrightarrow t_2 \prec t_2}\;\mathsf{ax}}{\longrightarrow t_2 \prec t_2}\;\mathsf{refl}}{p \,@\, t_2 \longrightarrow \Diamond p \,@\, t_2}\;r\Diamond \quad \dfrac{\dfrac{}{p \,@\, t_1 \longrightarrow p \,@\, t_1}\;\mathsf{ax} \quad \boxed{1}}{{}^{0 \prec t_1,}_{0 \prec t_2}\; p \,@\, t_1 \longrightarrow \Diamond p \,@\, t_2, t_1 \prec t_2}\;r\Diamond}{0 \prec t_1, 0 \prec t_2, p \,@\, t_1, \Box p \,@\, t_1 \longrightarrow \Diamond p \,@\, t_2}\;l\Box \quad \dfrac{\dfrac{}{t_1 \prec t_1 \longrightarrow t_1 \prec t_1}\;\mathsf{ax}}{\longrightarrow t_1 \prec t_1}\;\mathsf{refl}}{0 \prec t_1, 0 \prec t_2, \Box p \,@\, t_1 \longrightarrow \Diamond p \,@\, t_2}\;l\Box}{0 \prec t_1, \Box p \,@\, t_1 \longrightarrow \Box\Diamond p \,@\, 0}\;r\Box}{\Diamond\Box p \,@\, 0 \longrightarrow \Box\Diamond p \,@\, 0}\;l\Diamond}{\longrightarrow \Diamond\Box p \supset \Box\Diamond p \,@\, 0}\;r\supset$$

Bottom of the tree

Figure 6: a proof of axiom $\Diamond\Box p \supset \Box\Diamond p \,@\, 0$, characteristic of reflexive, weakly directed frames, in $\mathcal{C}_{\mathbf{QS4.3}}$ — but frame rules are explicity indicated. The bottom subtree is at the root of the proof; the subtrees above correspond to placeholders $\boxed{1}$ and $\boxed{2}$. Notice the difference with the proof in Figure 5, in which logical rules are used above rule wconn.

with the notation $t\!:\!\theta$ (if t denotes an element in $\mathcal{W}'$) or $t\!:\!\iota$ (if t denotes an element in $\mathcal{C}'$).

The *interpretation* I' maps function and predicate symbols to functions and predicates over $\mathcal{D}'$, respecting the rank of each symbol; in particular, it maps $=_\iota$ to the equality relation over $\mathcal{W}'$. An *assignment* in $2FOL$ is a function α' mapping variable symbols in $\mathcal{V}'$ to elements of either sort, depending on their rank. Given the standard notion of denotation of terms, truth of a $2FOL$ formula in $\mathbf{M}'$ under α' is the usual one for many-sorted first-order logics.

Definition 11 ($\mathcal{C}_{\mathbf{QK}}$) *Let A, B be formulae, c_1, c_2, s, t terms and a_1, a_2 variables of $2FOL$; then $2LK$, a sequent calculus for $2FOL$, is visible in Table 5.*

$2LK$ is a specialisation for two sorts of the calculus $G_=$ for many-sorted languages with equality (see [Gallier, 1986], Definition 10.5.1), where equality is admitted on one sort only, namely the sort ι; the presentation is also simplified with respect to Gallier's according to [Degtyarev and Voronkov, 2001, Kanger, 1983]. $2LK$ consists of an axiomatic rule, rules for equality, rules for $\neg$ and $\supset$, and two pairs of rules for $\forall$, denoted $r\forall^*_\iota, r\forall^*_\theta, l\forall^*_\iota, l\forall^*_\theta$, in place of the usual ones for untyped quantifiers. We denote all $2LK$-rules with a superscript $*$.

We have that

Theorem 12 *$2LK$ is sound and complete for $2FOL$ (Lemma 10.5.1, Theorem 10.5.1 in [Gallier, 1986]; Section 1 of [Degtyarev and Voronkov, 2001]).*

$$\dfrac{\dfrac{\dfrac{}{p @ t_2 \longrightarrow p @ t_2}\,\mathsf{ax}}{t_1 \doteq t_2, p @ t_1 \longrightarrow p @ t_2}\,\mathsf{sub}_{\doteq}}{la(0) \doteq t_1, la(0) \doteq t_2, p @ t_1 \longrightarrow p @ t_2}\,\mathsf{sub}_{\doteq}$$

Branch $\boxed{3}$

$$\dfrac{\dfrac{}{0 \prec la(0) \longrightarrow 0 \prec la(0)}\,\mathsf{ax} \qquad \dfrac{}{0 \prec la(0) \longrightarrow 0 \prec la(0)}\,\mathsf{ax}}{\longrightarrow 0 \prec la(0)}\,\mathsf{atom}$$

Branch $\boxed{2}$

$$\dfrac{\dfrac{}{0 \prec la(0) \longrightarrow 0 \prec la(0)}\,\mathsf{ax} \qquad \dfrac{}{0 \prec la(0) \longrightarrow 0 \prec la(0)}\,\mathsf{ax}}{\longrightarrow 0 \prec la(0)}\,\mathsf{atom}$$

Branch $\boxed{1}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\boxed{3} \qquad \dfrac{}{la(0) \prec t_1 \longrightarrow la(0) \prec t_1}\,\mathsf{ax}}{la(0) \prec t_1, la(0) \doteq t_1, p @ t_2 \longrightarrow p @ t_1}\,\mathsf{atom} \qquad \dfrac{}{la(0) \prec t_2 \longrightarrow la(0) \prec t_2}\,\mathsf{ax}}{la(0) \prec t_1, la(0) \prec t_2, p @ t_2 \longrightarrow p @ t_1}\,\mathsf{atom}}{la(0) \prec t_1, \Diamond p @ la(0) \longrightarrow p @ t_1}\,l\Diamond \qquad \boxed{2}}{la(0) \prec t_1, \Box\Diamond p @ 0 \longrightarrow p @ t_1}\,l\Box}{\Box\Diamond p @ 0 \longrightarrow \Box p @ la(0)}\,r\Box \qquad \boxed{1}}{\Box\Diamond p @ 0 \longrightarrow \Diamond\Box p @ 0}\,r\Diamond}{\longrightarrow \Box\Diamond p \supset \Diamond\Box p @ 0}\,r\supset$$

Bottom of the tree

Figure 7: a proof of McKinsey's axiom, $\Box\Diamond p \supset \Diamond\Box p @ 0$, in $\mathcal{C}_{\mathbf{QS4.1}}$. Notice that this proof is normal, as expected. The bottom subtree is at the root of the proof; the subtrees above correspond to placeholders $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$.

4.2 Embedding QMLs into $2FOL$

Now we define a translation which maps formulae and first-order sentences to terms and formulae of $2FOL$:

Definition 13 *Let the operator* $[\![\cdot]\!]$ *be defined as in Figure 8 (recall Definition 1).*

$[\![\cdot]\!]$ maps symbols of **QL** to primed symbols of $2FOL$, logical atoms to $2FOL$ atoms with one more argument (the label), and leaves other formulae as they stand, recursively, except that it respects the type of the quantifiers in $\forall$-formulae (which is always θ) and it unfolds $\Box$ operators in a way that is intuitively related to their semantics. As an example,

$$[\![\forall x.\Box p(x) \supset p(x) @ 0]\!] = \forall x' : \iota . \forall t' : \theta . 0' \prec' t' \supset p'(x', t') \supset p'(x', 0').$$

In the following, we drop the "prime" in order to ease the notation. The above example becomes

Closing rules

$$\frac{}{\Gamma, A \longrightarrow A, \Delta}\ ax^*$$

Rules for equality

$$\frac{}{\Gamma \longrightarrow \tau =_\iota \tau, \Delta}\ re^* \qquad \frac{\Gamma[t\!:\!\iota/x], s =_\iota t \longrightarrow \Delta[t\!:\!\iota/x]}{\Gamma[s\!:\!\iota/x], s =_\iota t \longrightarrow \Delta[s\!:\!\iota/x]}\ sub^*$$

Logical rules

$$\frac{\Gamma \longrightarrow A, \Delta}{\Gamma, \neg A \longrightarrow \Delta}\ l\neg^* \qquad \frac{\Gamma, A \longrightarrow \Delta}{\Gamma \longrightarrow \neg A, \Delta}\ r\neg^*$$

$$\frac{\Gamma, B \longrightarrow \Delta \quad \Gamma \longrightarrow A, \Delta}{\Gamma, A \supset B \longrightarrow \Delta}\ l\supset^* \qquad \frac{\Gamma, A \longrightarrow B, \Delta}{\Gamma \longrightarrow A \supset B, \Delta}\ r\supset^*$$

$$\frac{\Gamma, \forall x\!:\!\iota.A, A[c_1\!:\!\iota/x] \longrightarrow \Delta}{\Gamma, \forall x\!:\!\iota.A \longrightarrow \Delta}\ l\forall^*_\iota \qquad \frac{\Gamma \longrightarrow A[a_1\!:\!\iota/x], \Delta}{\Gamma \longrightarrow \forall x\!:\!\iota.A, \Delta}\ r\forall^*_\iota$$

$$\frac{\Gamma, \forall x\!:\!\theta.A, A[c_2\!:\!\theta/x] \longrightarrow \Delta}{\Gamma, \forall x\!:\!\theta.A \longrightarrow \Delta}\ l\forall^*_\theta \qquad \frac{\Gamma \longrightarrow A[a_2\!:\!\theta/x], \Delta}{\Gamma \longrightarrow \forall x\!:\!\theta.A, \Delta}\ r\forall^*_\theta$$

Table 5: the calculus $2LK$ for $2FOL$. A, B are formulae, c_1, c_2, s, t terms and a_1, a_2 variables of $2FOL$; a_1 and a_2 cannot appear free in the conclusion of $r\forall^*_\iota$ and $r\forall^*_\theta$. In rule sub^*, the occurrences of $s\!:\!\iota$ replaced by $t\!:\!\iota$ are in atomic formulae only.

$$[\![\forall x.\Box p(x) \supset p(x) \ @\ 0]\!] = \forall x\!:\!\iota.\forall t\!:\!\theta.\ 0 \prec t \supset p(x,t) \supset p(x,0).$$

$[\![\cdot]\!]$ is also straightforwardly extended to sequent rules:

$$\frac{\Gamma_1 \longrightarrow \Delta_1 \quad \cdots \quad \Gamma_n \longrightarrow \Delta_n}{\Gamma \longrightarrow \Delta}\ \rho \qquad \overset{[\![\cdot]\!]}{\rightsquigarrow} \qquad \frac{[\![\Gamma_1 \longrightarrow \Delta_1]\!] \quad \cdots \quad [\![\Gamma_n \longrightarrow \Delta_n]\!]}{[\![\Gamma \longrightarrow \Delta]\!]}\ [\![\rho]\!]$$

As it can be seen, $[\![\cdot]\!]$ preserves the number of premises of a sequent rule; therefore, it extends also to derivations: the $2FOL$-translation of a derivation is a derivation in which all sequent rules are $2FOL$-translations of sequent rules of $\mathcal{C}_{\mathbf{QL}}$. The same goes for proofs.

4.3 Soundness and completeness

For any $\mathbf{QL}$, let $\mathsf{FrmAx}^S(\mathbf{QL})$ be the set of first-order sentences obtained by skolemising and converting in prenex normal forms the sentences in $\mathsf{FrmAx}(\mathbf{QL})$. Let also, as usual, Γ and Δ be finite multisets of $\mathbf{forms}$, with the restriction that the labels appearing in $\Gamma \cup \Delta$ must not contain any Skolem functions. In order to show soundness and completeness of $\mathcal{C}_{\mathbf{QL}}$ for $\mathbf{QL}$, we prove that the following statements are equivalent:

1. $\Gamma \longrightarrow \Delta$ is a theorem of $\mathcal{C}_{\mathbf{QL}}$,
2. $[\![\Gamma \cup \mathsf{FrmAx}^S(\mathbf{QL}) \longrightarrow \Delta]\!]$ is a theorem of $2LK$,
3. $[\![\Gamma \cup \mathsf{FrmAx}(\mathbf{QL}) \longrightarrow \Delta]\!]$ is valid in $2FOL$,

$[\![x]\!]$, with $x \in \mathcal{V}$	$=$	$x' : \iota \in \mathcal{V}'$
$[\![f]\!]$, with $f \in \mathcal{F}$	$=$	$f' \in \mathcal{F}'$
$[\![p]\!]$, with $p \in \mathcal{P}$	$=$	$p' \in \mathcal{P}'$
$[\![0]\!]$	$=$	$0' \in \mathcal{F}'$
$[\![\prec]\!]$	$=$	$\prec' \in \mathcal{P}'$
$[\![\dot{=}]\!]$	$=$	$=_\iota \in \mathcal{P}'$
$[\![t]\!]$, with $t \in \mathcal{V}_t$	$=$	$t' : \theta \in \mathcal{V}'$
$[\![g]\!]$, with $g \in \mathcal{F}_t$	$=$	$g' \in \mathcal{F}'$
$[\![\tau_1 \prec \tau_2]\!]$	$=$	$[\![\tau_1]\!] \prec' [\![\tau_2]\!]$
$[\![p(s_1, \ldots, s_n) @ \tau]\!]$	$=$	$[\![p]\!]([\![s_1]\!], \ldots, [\![s_n]\!], [\![\tau]\!])$
$[\![\neg\varphi @ \tau]\!]$	$=$	$\neg[\![\varphi @ \tau]\!]$
$[\![\varphi \supset \psi @ \tau]\!]$	$=$	$[\![\varphi @ \tau]\!] \supset [\![\psi @ \tau]\!]$
$[\![\forall x.\varphi @ \tau]\!]$	$=$	$\forall x : \iota.\ [\![\varphi @ \tau]\!]$
$[\![\Box\varphi @ \tau]\!]$	$=$	$\forall t : \theta.\ [\![\tau \prec t]\!] \supset [\![\varphi @ t]\!]$
$[\![\Gamma]\!]$	$=$	$\{[\![\gamma]\!] \mid \gamma \in \Gamma\}$
$[\![\Gamma \longrightarrow \Delta]\!]$	$=$	$[\![\Gamma]\!] \longrightarrow [\![\Delta]\!]$
$[\![\phi]\!]$, ϕ a first-order sentence	$=$	ϕ

Figure 8: the definition of $[\![\cdot]\!]$, a $2FOL$-translation mapping formulae, sequents and first-order sentences to formulae and sequents of $2FOL$. Translations of first-order sentences ϕ include the rank of bound variables, which is invariably θ.

4. $\Gamma \longrightarrow \Delta$ is valid in $\mathbf{QL}$.

Figure 9 graphically depicts the situation.

$$\begin{array}{ccccc}
\vdash_{\mathcal{C}_{\mathbf{QL}}} \Gamma \longrightarrow \Delta \quad \boxed{1} & \Longleftrightarrow & \boxed{4} \quad \models^{\mathbf{QL}} \Gamma \longrightarrow \Delta \\
\Updownarrow & & \Updownarrow \\
\vdash_{2LK} [\![\Gamma \cup \mathsf{FrmAx}^S(\mathbf{QL}) \longrightarrow \Delta]\!] \quad \boxed{2} & \Longleftrightarrow & \boxed{3} \quad \models^{2FOL} [\![\Gamma \cup \mathsf{FrmAx}(\mathbf{QL}) \longrightarrow \Delta]\!]
\end{array}$$

Figure 9: a schematic representation of the proof of correctness. Instead of proving that 1 implies 4 (soundness) and that 4 implies 1 (completeness), we prove that 1, 2, 3 and 4 are equivalent.

We proceed by first proving equivalence 1-2 (Proposition 14), then equivalence 2-3 (Proposition 25), and lastly equivalence 3-4 (Proposition 26).

Proposition 14 (Equivalence 1-2) *Items 1 and 2 are equivalent, that is*

$$\vdash_{\mathcal{C}_{\mathbf{QL}}} \Gamma \longrightarrow \Delta \quad \textit{iff} \quad \vdash_{2LK} [\![\Gamma \cup \mathsf{FrmAx}^S(\mathbf{QL}) \longrightarrow \Delta]\!].$$

Proof: we show that every $\mathcal{C}_{\mathbf{QL}}$-proof can be $2FOL$-translated, and that every $2LK$-proof of a $2FOL$-translated sequent is similar to a $2LK$-proof that is the $2FOL$-translation of a $\mathcal{C}_{\mathbf{QL}}$-proof.

1 implies 2: in order to show that every $\mathcal{C}_{\mathbf{QL}}$-proof can be $2FOL$-translated, we show that every rule in $\mathcal{C}_{\mathbf{QL}}$ can be $2FOL$-translated. Recall Tables 2, 3 and 5; by case analysis,

- *closing rule:* straightforwardly, $[\![\mathsf{ax}]\!] = ax^*$.

- *logical rules:* as well, $[\![r\neg]\!] = r\neg^*$, $[\![l\neg]\!] = l\neg^*$, $[\![r\supset]\!] = r\supset^*$ and $[\![l\supset]\!] = l\supset^*$. For example:

$$\frac{\Gamma, \varphi \ @\ \tau \longrightarrow \psi \ @\ \tau, \Delta}{\Gamma \longrightarrow \varphi \supset \psi \ @\ \tau, \Delta}\ r\supset \qquad \overset{[\![\cdot]\!]}{\rightsquigarrow} \qquad \frac{[\![\Gamma]\!], [\![\varphi \ @\ \tau]\!] \longrightarrow [\![\psi \ @\ \tau]\!], [\![\Delta]\!]}{[\![\Gamma]\!] \longrightarrow [\![\varphi \ @\ \tau]\!] \supset [\![\psi \ @\ \tau]\!], [\![\Delta]\!]}\ r\supset^*$$

Moreover, $[\![r\forall]\!] = r\forall_\iota^*$ and $[\![l\forall]\!] = l\forall_\iota^*$. For example:

$$\frac{\Gamma \longrightarrow \varphi[a/x] \ @\ \tau, \Delta}{\Gamma \longrightarrow \forall x.\varphi \ @\ \tau, \Delta}\ r\forall \qquad \overset{[\![\cdot]\!]}{\rightsquigarrow} \qquad \frac{[\![\Gamma]\!] \longrightarrow [\![\varphi[a/x] \ @\ \tau]\!], [\![\Delta]\!]}{[\![\Gamma]\!] \longrightarrow \forall x.[\![\varphi \ @\ \tau]\!], [\![\Delta]\!]}\ r\forall_\iota$$

Lastly, $[\![r\Box]\!]$ is the composition of $r\forall_\theta^*$ and $r\supset^*$, whereas $[\![l\Box]\!]$ is the composition of $l\forall_\theta^*$ and $l\supset^*$. For example:

$$\frac{\Gamma, \tau \prec t_a \longrightarrow \varphi \ @\ t_a, \Delta}{\Gamma \longrightarrow \Box\varphi \ @\ \tau, \Delta}\ r\Box \qquad \overset{[\![\cdot]\!]}{\rightsquigarrow} \qquad \frac{\dfrac{[\![\Gamma]\!], [\![\tau \prec t_a]\!] \longrightarrow [\![\varphi \ @\ t_a]\!], [\![\Delta]\!]}{[\![\Gamma]\!] \longrightarrow [\![\tau \prec t_a]\!] \supset [\![\varphi \ @\ t_a]\!], [\![\Delta]\!]}\ r\supset^*}{[\![\Gamma]\!] \longrightarrow \forall t{:}\theta.[\![\tau \prec t]\!] \supset [\![\varphi \ @\ t]\!], [\![\Delta]\!]}\ r\forall_\theta^*$$

- *frame rules:* again, $[\![\mathsf{refl}_{\doteq}]\!] = re^*$, $[\![\mathsf{sub}_{\doteq}]\!] = sub^*$ and, by Definition 6, frame rules obtained by the strengthening procedure are finite compositions of $2LK$ rules.

This completes the proof of implication 1-2.

2 implies 1: this case is more complicated. From now on, let $[\![\mathbf{forms}]\!]$ denote the image of $\mathbf{forms}$ under $[\![\cdot]\!]$, that is $[\![\mathbf{forms}]\!] = \{\psi \mid \psi = [\![\varphi]\!], \varphi \in \mathbf{forms}\}$; moreover, let any $2FOL$-formula which is the translation of a formula $\varphi \in \mathbf{forms}$ be denoted as $[\![\varphi]\!]$; lastly, let us assume that Π is a $2LK$-proof of $[\![\Gamma \cup \mathsf{FrmAx}^S(\mathbf{QL}) \longrightarrow \Delta]\!]$, for some logic $\mathbf{QL}$ and Γ, Δ multisets of $\mathbf{forms}$.

We want to show that there is a $2LK$-proof Π', similar to Π, which is the translation of a $\mathcal{C}_{\mathbf{QL}}$-proof of $\Gamma \longrightarrow \Delta$. In order to do that, we first establish a sufficient condition for a $2LK$-proof to be the translation of a $\mathcal{C}_{\mathbf{QL}}$-proof, and then we show that, for every Π, there is a similar Π' which enjoys the condition.

A *subset* of Π is a subset of the nodes of Π; let $N \in \Pi$ be labelled by $[\![\varphi]\!]$; then

Definition 15 *A* trail *of N, $\mathsf{Tr}(N)$, is a subset of Π for which the following properties hold:*

1. *$\mathsf{Tr}(N)$ is a tree and N is the root node;*
2. *let $N_i \in \mathsf{Tr}(N)$; let $N_j, j = 1 \ldots, n$ be its children, each one labelled by $[\![\varphi_j]\!]$; then every $[\![\varphi_j]\!]$ is active in N_i;*
3. *no node of the trail is labelled by a duplicate $\forall$-formula introduced in the premises by a $l\forall_\theta^*$ rule.*

$\mathsf{Tr}(N)$ *is said to* belong *to Π, which is said to be its* parent*.*

Informally speaking, the trail of N is the subset of Π by which $[\![\varphi]\!]$ is "completely unfolded".

Let $(N_1, \ldots, N_k)$ be a branch of Π; then a *path* in Π is a tuple of nodes $(N_n, \ldots, N_m)$ such that $1 \leq n \leq m \leq k$, and its *length*, $len(N_n, \ldots, N_m)$, is the number of nodes between N_n and N_m. The *sparsity* of a trail $\mathsf{Tr}(N)$ in Π is defined as $\sum len(N', \ldots, N'')$ for all $N', N'' \in \Pi$ such that N'' is a child of N' in $\mathsf{Tr}(N)$. Intuitively, the sparsity of a trail indicates how "far away" from each other the nodes of $\mathsf{Tr}(N)$ are in its parent. If the sparsity is 0, the trail is called *compact*. Informally speaking, a compact trail is also a proper subtree of its parent.

Definition 16 (Compactness of a proof) *A $2LK$-proof Π will be called* compact *if and only if:*

1. *for every node $N \in \Pi$ labelled by $[\![\varphi]\!]$, $\mathsf{Tr}(N)$ belongs to Π and is compact;*
2. *the union of all such trails is Π.*

Informally, a compact proof is the union of a finite set of compact trails, each one labelled by a $2FOL$-formula $[\![\varphi]\!]$. We are now ready to prove that the property of compactness is sufficient for a $2FOL$-proof to be the translation of a $\mathcal{C}_{\mathbf{QL}}$-proof:

Lemma 17 *If Π is compact, then there is a $\mathcal{C}_{\mathbf{QL}}$-proof Θ such that $[\![\Theta]\!] = \Pi$.*

Proof: immediate from Definition 15 and the proof of implication 1-2: every translation of a $\mathcal{C}_{\mathbf{QL}}$-rule ρ is the compact trail of a node N in a $2LK$-proof, labelled by ρ itself. •

Thanks to this Lemma, in order to prove implication 2-1, it suffices to show that for every Π there is a similar, compact Π'. To carry on, we first need two useful results from Proof Theory:

Lemma 18 (Inversion Lemma for $2LK$) *For all $\rho \in 2LK$ except ax^* and re^*, if the conclusion of ρ is $2LK$-provable, so are all the premises.*

Proof: by induction on the depth of a proof, that is, on the length of the longest branch in the proof. See Proposition 3.5.4 in [Troelstra and Schwichtenberg, 1996] for the details. The Proposition also trivially extends to rule sub^*. •

Given the notions of *adjacency* and *permutability* of sequent rules in $2LK$, adapted from Definition 5.3.1 in [Troelstra and Schwichtenberg, 1996],

Lemma 19 (Permutation Lemma for $2LK$) *Let $\rho, \rho' \in 2LK$. Then ρ is always permutable below ρ', except when $\rho = l\forall^*_\iota$ and $\rho' = r\forall^*_\iota$, or when $\rho = l\forall^*_\theta$ and $\rho' = r\forall^*_\theta$. The new proof is similar to the original one.*

Proof: as in Lemma 5.3.10 in [Troelstra and Schwichtenberg, 1996], specialised for two sorts and no structural rules. The definition of permutability obviously takes into account the fact that no rule is permutable where it is not applicable, i.e., that rule α can be permuted below rule β only if the main formula in α is not active in β and vice-versa. •

Now we proceed by case analysis on the shape of $[\![\Gamma \cup \mathsf{FrmAx}^S(\mathbf{QL}) \longrightarrow \Delta]\!]$, considering in turn three subcases, and showing that in each (more and more complex) subcase, there is Π' which is similar to Π and compact.

Subcase (I) Let the set of subformulae of $\Gamma \cup \Delta$ contain no $\Box$-formulae and let $\mathsf{FrmAx}^S(\mathbf{QL})$ be empty. By structural induction on the shape of the subformulae of $[\![\Gamma]\!]$ and $[\![\Delta]\!]$, it is clear that every node $N \in \Pi$ is labelled by a $2LK$-rule displayed in Table 5, *except* $r\forall^*_\theta$ and $l\forall^*_\theta$.

But, each of these rules is the translation of a single $\mathcal{C}_{\mathbf{QL}}$-rule (recall the proof of implication 1-2); therefore, by Definition 15, every node in Π is a single, compact trail. Then Π is compact by Definition 16, and obviously similar to itself.

Subcase (II) Suppose now that there is at least a node $N \in \Pi$ labelled by a $\Box$-formula. We first state a corollary of Lemma 19:

Corollary 20 *An application of rule $r{\supset}^*$ or $l{\supset}^*$ can be permuted below or above any other rule, preserving similarity.*

Let us call a $\Box$-*trail* the trail of a node N labelled by the translation of a $\Box$-formula; then

Theorem 21 (Existence and compactness of $\Box$-trails) *Let $N \in \Pi$ be labelled by the translation of a $\Box$-formula; then there is a $2LK$-proof Π' similar to Π such that:*

1. $\mathsf{Tr}(N)$ *belongs to* Π',
2. $\mathsf{Tr}(N)$ *is compact.*

Proof: *(1)*: by contradiction. Consider node N: by the conclusion of $r\forall^*_\theta$ we know that

$$\vdash_{2LK} \Gamma \longrightarrow \forall t\!:\!\theta.[\![\tau \prec t]\!] \supset [\![\varphi \mathbin{@} t]\!], \Delta.$$

Now if *(1)* is false, then by Definition 15, there can be no proof Π' in which a node N' above N is labelled by $r{\supset}^*$, and its main formula is active in N. This means that

$$\not\vdash_{2LK} \Gamma \longrightarrow [\![\tau \prec a]\!] \supset [\![\varphi \mathbin{@} a]\!], \Delta$$

where a does not appear free in the former sequent. But this contradicts Lemma 18, when $\rho = r\forall^*_\theta$. An analogous argument holds on the left. *(2)*: By Corollary 20, the child of N in $\mathsf{Tr}(N)$ can be permuted in Π so that the sparsity of $\mathsf{Tr}(N)$ eventually becomes 0, that is, $\mathsf{Tr}(N)$ is compact. •

Let then Π' be such a proof: by this very Theorem, all $\Box$-trails in Π belong to Π', and they are all compact. Moreover, as it can be easily checked, Π' does not contain any new nodes labelled by $\Box$-formulae; and, since by the same inductive argument of Subcase I, the only nodes in Π' not falling in the previous Subcase are exactly those in all $\Box$-trails, all nodes in Π' belong to a compact trail. By Definition 16 then, Π' is compact, and it is similar to Π by this Theorem again.

As an example, let Π be the following $2LK$-proof of theorem $[\![\Box(p \vee \neg p) \mathbin{@} 0]\!]$ (all bound variables have sort θ — we omit it for the sake of conciseness):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{[\![\Gamma']\!], 0 \prec t_a, p(t_a) \longrightarrow p(t_a), [\![\Delta']\!]}\;ax^*}{[\![\Gamma']\!], 0 \prec t_a \longrightarrow p(t_a), \neg p(t_a), [\![\Delta']\!]}\;r\neg^*}{[\![\Gamma']\!], 0 \prec t_a \longrightarrow p(t_a) \vee \neg p(t_a), [\![\Delta']\!]}\;r\vee^*}{[\![\Gamma']\!] \longrightarrow 0 \prec t_a \supset p(t_a) \vee \neg p(t_a), [\![\Delta']\!]}\;r{\supset}^*$$

$$\vdots\ \text{subproof \#1}$$

$$\dfrac{[\![\Gamma]\!] \longrightarrow 0 \prec t_a \supset p(t_a) \vee \neg p(t_a), [\![\Delta]\!]}{[\![\Gamma]\!] \longrightarrow \forall t. 0 \prec t \supset p(t) \vee \neg p(t), [\![\Delta]\!]}\;r\forall^*_\theta$$

Assume, withouth loss of generality, that subproof #1 is compact, and let Π' be the following proof:

$$\dfrac{\dfrac{\dfrac{}{[\![\Gamma']\!], 0 \prec t_a, p'(t_a) \longrightarrow p'(t_a), [\![\Delta']\!]}\;ax^*}{[\![\Gamma']\!], 0 \prec t_a \longrightarrow p'(t_a), \neg p'(t_a), [\![\Delta']\!]}\;r\neg^*}{[\![\Gamma']\!], 0 \prec t_a \longrightarrow p'(t_a) \vee \neg p'(t_a), [\![\Delta']\!]}\;r\vee^*$$

$$\vdots\ \text{subproof \#1}$$

$$\dfrac{\dfrac{[\![\Gamma]\!], 0 \prec t_a \longrightarrow p'(t_a) \vee \neg p'(t_a), [\![\Delta]\!]}{[\![\Gamma]\!] \longrightarrow 0 \prec t_a \supset p'(t_a) \vee \neg p'(t_a), [\![\Delta]\!]}\;r{\supset}^*}{[\![\Gamma]\!] \longrightarrow \forall t. 0 \prec t \supset p'(t) \vee \neg p'(t), [\![\Delta]\!]}\;r\forall^*_\theta$$

It turns out that $\Pi' = [\![\Theta]\!]$, where Θ is the following $\mathcal{C}_{\mathbf{QK}}$-proof:

$$\dfrac{\dfrac{\dfrac{}{\Gamma', 0 \prec t_a, p \mathbin{@} t_a \longrightarrow p \mathbin{@} t_a, \Delta'}\;\mathsf{ax}}{\Gamma', 0 \prec t_a \longrightarrow p \mathbin{@} t_a, \neg p \mathbin{@} t_a, \Delta'}\;r\neg}{\Gamma', 0 \prec t_a \longrightarrow p \vee \neg p \mathbin{@} t_a, \Delta'}\;r\vee$$

$$\vdots\ \text{subproof \#1}$$

$$\dfrac{\Gamma, 0 \prec t_a \longrightarrow p \vee \neg p \mathbin{@} t_a, \Delta}{\Gamma \longrightarrow \Box(p \vee \neg p) \mathbin{@} 0, \Delta}\;r\Box$$

and that Π' is similar to Π' and compact, as we expect from Theorem 21.

Subcase (III) Suppose, lastly, that there is at least $N \in \Pi$ labelled by $\phi^S \in \mathsf{FrmAx}^S(\mathbf{QL})$. Another immediate corollary of Lemma 19 is that

Corollary 22 *An application of rule $l\forall^*_\theta$, $l\neg^*$, $r\neg^*$, $l\supset^*$, $r\supset^*$ can be permuted* above *any other rule, preserving similarity.*

Let us call a *frame trail* $\mathsf{Tr}(N)$, where N is labelled by the translation of ϕ^S; then

Theorem 23 (Existence and compactness of frame trails) *Suppose $N \in \Pi$ is labelled by the translation of a frame axiom ϕ^S; then there is a $2LK$-proof Π' similar to Π such that:*

1. $\mathsf{Tr}(N)$ *belongs to* Π',
2. $\mathsf{Tr}(N)$ *is compact.*

Proof: *(1)*: by the same argument of Theorem 21 and repeated application of Lemma 18. *(2)*: by the same argument of Theorem 21: by Corollary 22, and by the fact that by Definition 6, ϕ^S has the shape $\forall\overline{x}.P(\overline{x})$, where $P(\overline{x})$ is quantifier-free and appears on the left of a sequent. •

Let then Π' be such a proof: by this very Corollary, all frame trails in Π belong to Π', and they are all compact. Moreover, again, Π' does not contain any new nodes labelled by frame axioms.

However here, differently from the previous Subcase, the use of the $l\forall^*_\theta$ rule *can* spawn nodes which do not belong to any trail; in fact, by the same inductive argument of Subcases I and II, this is the only case of nodes in Π' not falling in the previous Subcases. So it remains to prove that there is a further $2LK$-proof, call it Π'', similar to Π and Π', in which such nodes belong to a compact trail.

As an example of "bad" behaviour, consider Figure 10, illustrating a proof involving the axiom of symmetry (indicated as 5 to ease the notation — recall Table 1). The problem arises from the very shape of frame axioms, which can have, in general, two or more outer universal quantifiers.

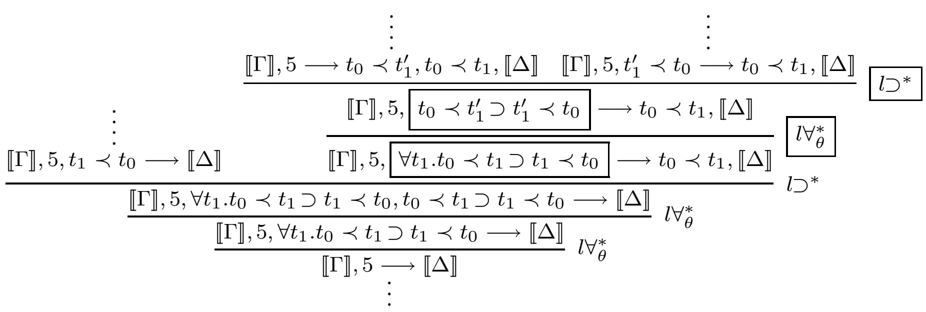

Figure 10: an example of "bad" frame trail: an applicaton of rule $l\forall^*_\theta$, boxed in the Figure, generates a duplicate $\forall$-formula which is not the translation of any formula in $\mathbf{forms}$ and spawns nodes not belonging to any trail. Bad nodes are boxed, as well as their main formulae.

Let $N' \in \Pi$ be labelled by a duplicate formula ψ. It must be the case that ψ was generated by a $l\forall^*_\theta$ labelling a node in a frame trail; call the frame axiom at the root of the trail ϕ^S. Now since ϕ^S is in prenex normal form, it must be the case that $\phi^S = \forall x_1 \ldots x_n.\psi$, and that there is a copy of ϕ^S in the sequent labelling N'. This is evident in Figure 10, at the node labelled by the boxed $l\forall^*_\theta$.

Let then $N_1, \ldots, N_n$ be n new nodes inserted just below N', such that *(a)* N_1 is labelled by ϕ^S and $l\forall^*_\theta$, *(b)* for all $N_i, i = 1, \ldots, n$, N_i is labelled by $\forall x_i \ldots x_n.\psi$ and $l\forall^*_\theta$. Let, lastly, N' be labelled by the active formula in N_n. This way we obtain a new proof Π'' similar to Π' which contains a trail $\mathsf{Tr}(N_1)$ labelled by ϕ^S including the old bad nodes.

Figure 11 shows the effect of this operation on the example of Figure 10.

By repeated application of this method, all nodes not falling in the above cases can be revamped into nodes belonging to frame trails; more formally, there is Π'' similar to Π which meets Definition 16 and is therefore compact.

In order to carry the proof of implication 2-1 to the end, one last simple result is needed:

$$
\dfrac{\vdots \qquad \dfrac{\dfrac{\dfrac{\vdots \quad\quad \vdots}{\llbracket\Gamma\rrbracket, 5 \longrightarrow t_0 \prec t_1', t_0 \prec t_1, \llbracket\Delta\rrbracket \quad \llbracket\Gamma\rrbracket, 5, t_1' \prec t_0 \longrightarrow t_0 \prec t_1, \llbracket\Delta\rrbracket}}{\llbracket\Gamma\rrbracket, 5, \boxed{t_0 \prec t_1' \supset t_1' \prec t_0} \longrightarrow t_0 \prec t_1, \llbracket\Delta\rrbracket}\ \boxed{l\supset^*}}{\llbracket\Gamma\rrbracket, 5, \boxed{\forall t_1 . t_0 \prec t_1 \supset t_1 \prec t_0} \longrightarrow t_0 \prec t_1, \llbracket\Delta\rrbracket}\ \boxed{l\forall^*_\theta}}{\llbracket\Gamma\rrbracket, 5, t_1 \prec t_0 \longrightarrow \llbracket\Delta\rrbracket \qquad \llbracket\Gamma\rrbracket, \boxed{5} \longrightarrow t_0 \prec t_1, \llbracket\Delta\rrbracket}\ \boxed{l\forall^*_\theta}
$$

$$
\dfrac{\dfrac{\dfrac{\llbracket\Gamma\rrbracket, 5, t_1 \prec t_0 \longrightarrow \llbracket\Delta\rrbracket \qquad \llbracket\Gamma\rrbracket, \boxed{5} \longrightarrow t_0 \prec t_1, \llbracket\Delta\rrbracket}{\llbracket\Gamma\rrbracket, 5, \forall t_1 . t_0 \prec t_1 \supset t_1 \prec t_0, t_0 \prec t_1 \supset t_1 \prec t_0 \longrightarrow \llbracket\Delta\rrbracket}\ l\supset^*}{\llbracket\Gamma\rrbracket, 5, \forall t_1 . t_0 \prec t_1 \supset t_1 \prec t_0 \longrightarrow \llbracket\Delta\rrbracket}\ l\forall^*_\theta}{\llbracket\Gamma\rrbracket, 5 \longrightarrow \llbracket\Delta\rrbracket}\ l\forall^*_\theta
$$

$$\vdots$$

Figure 11: the example of Figure 10, “cured”: a new node has been inserted in the proof, making the old bad nodes part of a new frame trail.

Lemma 24 *A rule ρ whose active formulae are atomic can be permuted above until it is at the top of the proof tree.*

Proof: by the definition of permutability (Definition 5.3.1 in [Troelstra and Schwichtenberg, 1996]), a rule ρ is permutable above a rule ρ' only if none of the active formulae of ρ is main in ρ'. But the only rule in which the main formulae are atomic is ax^* which is closing, and therefore is at the top of the proof tree. •

By applying this result repeatedly to all compact frame trails in Π', we get a final $2LK$-proof in which all frame trails either appear at the top of the proof tree or have frame trails above them. By Definition 9, such a proof is the translation of a $\mathcal{C}_{\mathbf{QL}}$-proof.

This also holds for the frame rule $\mathsf{sub}_{\doteq}$, and it is precisely the reason for the restriction on its application (recall Table 5).

This completes the proof of implication 2-1 and therefore of Proposition 14.

As a final example, let Π be the following $2LK$-proof in which the axiom of symmetry 5 (recall Table 1) has been employed:

$$
\dfrac{\overline{\llbracket\Gamma'''\rrbracket, t_1 \prec t_0 \longrightarrow \llbracket\Delta'''\rrbracket}\ ax^* \qquad \overline{\llbracket\Gamma'''\rrbracket \longrightarrow t_0 \prec t_1, \llbracket\Delta'''\rrbracket}\ ax^*}{}
$$

$$
\vdots\ \text{subproof \#4} \qquad\qquad \vdots\ \text{subproof \#5}
$$

$$
\dfrac{\llbracket\Gamma''\rrbracket, t_1 \prec t_0 \longrightarrow \llbracket\Delta''\rrbracket \qquad \llbracket\Gamma''\rrbracket \longrightarrow t_0 \prec t_1, \llbracket\Delta''\rrbracket}{\llbracket\Gamma''\rrbracket, t_0 \prec t_1 \supset t_1 \prec t_0 \longrightarrow \llbracket\Delta''\rrbracket}\ l\supset^*
$$

$$
\vdots\ \text{subproof \#3}
$$

$$
\dfrac{\llbracket\Gamma'\rrbracket, t_0 \prec t_1 \supset t_1 \prec t_0 \longrightarrow \llbracket\Delta'\rrbracket}{\llbracket\Gamma'\rrbracket, \forall t_1 . t_0 \prec t_1 \supset t_1 \prec t_0 \longrightarrow \llbracket\Delta'\rrbracket}\ l\forall^*_\theta
$$

$$
\vdots\ \text{subproof \#2}
$$

$$
\dfrac{\llbracket\Gamma\rrbracket, \forall t_1 . t_0 \prec t_1 \supset t_1 \prec t_0 \longrightarrow \llbracket\Delta\rrbracket}{\llbracket\Gamma\rrbracket, \forall t_0 t_1 . t_0 \prec t_1 \supset t_1 \prec t_0 \longrightarrow \llbracket\Delta\rrbracket}\ l\forall^*_\theta
$$

$$
\vdots\ \text{subproof \#1}
$$

Let Π' be the following proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{[\![\Gamma''']\!], t_1 \prec t_0 \longrightarrow [\![\Delta''']\!]}\ ax^* \qquad \dfrac{}{[\![\Gamma''']\!] \longrightarrow t_0 \prec t_1, [\![\Delta''']\!]}\ ax^*}{[\![\Gamma''']\!], t_0 \prec t_1 \supset t_1 \prec t_0 \longrightarrow [\![\Delta''']\!]}\ l{\supset}^*}{[\![\Gamma''']\!], \forall t_1 . t_0 \prec t_1 \supset t_1 \prec t_0 \longrightarrow [\![\Delta''']\!]}\ l\forall^*_\theta}{[\![\Gamma''']\!], \forall t_0 t_1 . t_0 \prec t_1 \supset t_1 \prec t_0 \longrightarrow [\![\Delta''']\!]}\ l\forall^*_\theta
$$

⋮ subproof #4/#5
⋮ subproof #3
⋮ subproof #2
⋮ subproof #1

It turns out that $\Pi' = [\![\Theta]\!]$, where Θ is the following $\mathcal{C}_{\mathbf{QK}}$-proof:

$$
\dfrac{\dfrac{}{\Gamma''', t_1 \prec t_0 \longrightarrow \Delta'''}\ \mathsf{ax} \qquad \dfrac{}{\Gamma''' \longrightarrow t_0 \prec t_1, \Delta'''}\ \mathsf{ax}}{\Gamma''' \longrightarrow \Delta'''}\ \mathsf{symm}
$$

⋮ subproof #4/#5
⋮ subproof #3
⋮ subproof #2
⋮ subproof #1

and that Π' is similar to Π, it is compact, and the only displayed frame rule appears at the top of the tree, as we expect from Theorem 23 and Corollary 24. •

Proposition 25 (Equivalence 2-3) *Items 2 and 3 are equivalent, that is*

$$\vdash_{2LK} [\![\Gamma \cup \mathsf{FrmAx}^S(\mathbf{QL}) \longrightarrow \Delta]\!] \quad \textit{iff} \quad \models^{2FOL} [\![\Gamma \cup \mathsf{FrmAx}(\mathbf{QL}) \longrightarrow \Delta]\!].$$

Proof: since $[\![\mathbf{forms}]\!] \cup \mathsf{FrmAx}^S(\mathbf{QL})$ is a strict subset of the formulae of $2FOL$, this equivalence follows from Theorem 12, with the remark that the $2FOL$ theory of $\mathsf{FrmAx}^S(\mathbf{QL})$ is a conservative extension of that of $\mathsf{FrmAx}(\mathbf{QL})$ (see, e.g., [Shoenfield, 1970], p. 55). •

Proposition 26 (Equivalence 3-4) *Items 3 and 4 are equivalent, that is*

$$\models^{2FOL} [\![\Gamma \cup \mathsf{FrmAx}(\mathbf{QL}) \longrightarrow \Delta]\!] \quad \textit{if and only if} \quad \models^{\mathbf{QL}} \Gamma \longrightarrow \Delta.$$

Proof: since $[\![\cdot]\!]$ extends to sequents straightforwardly, it suffices to prove the Proposition for single formulae. The Proposition is proved by showing that, given a model in $\mathbf{QL}$ for $\varphi \in \mathbf{forms}$, there is a corresponding model for $[\![\varphi]\!]$ in $2FOL$, and vice-versa.

Let $\mathbf{M} = \langle \mathcal{W}, R, \mathcal{D}, I \rangle$ and α be a structure and an assignment of $\mathbf{QL}$, and let $\mathbf{M}' = \langle \mathcal{D}', I' \rangle$ and α' be a structure and an assignment of $2FOL$ such that:

1. $\mathcal{D}' = \mathcal{W} \cup \mathcal{D}$,

2. I' interprets $\prec'$ as R and $\doteq$ as $=_\iota$,

3. for any predicate symbol $p \in \mathcal{P}$,

$$(s_1^{\mathbf{M},\alpha}, \ldots, s_n^{\mathbf{M},\alpha}) \in I(w, p) \ \text{ iff } \ (s_1^{\mathbf{M}',\alpha'}, \ldots, s_n^{\mathbf{M}',\alpha'}, \tau^{\mathbf{M}',\alpha'}) \in I'(p')$$

4. $\alpha'(v\!:\!\iota) = d' \in \mathcal{D}'$ iff $\alpha(v) = d \in \mathcal{D}$,

5. $\alpha'(t\!:\!\theta) = w' \in \mathcal{D}'$ iff $\alpha(t) = w \in \mathcal{W}$.

It turns out that $\mathbf{M}, \alpha \models \varphi$ iff $\mathbf{M}', \alpha' \models [\![\varphi]\!]$. This is proved by structural induction. Base cases:

- logical atoms: $\mathbf{M}, \alpha \models p(s_1, \ldots, s_n) @ \tau$ if and only if $(s_1^{\mathbf{M},\alpha}, \ldots, s_n^{\mathbf{M},\alpha}) \in I(w, p)$ if and only if $(s_1^{\mathbf{M}',\alpha'}, \ldots, s_n^{\mathbf{M}',\alpha'}, \tau^{\mathbf{M}',\alpha'}) \in I'(p')$ if and only if $\mathbf{M}', \alpha' \models p'([\![s_1]\!], \ldots, [\![s_n]\!], [\![\tau]\!])$ that is $\mathbf{M}', \alpha' \models [\![p(s_1, \ldots, s_n) @ \tau]\!]$.
- $\prec$-constraints: $\mathbf{M}, \alpha \models \tau_1 \prec \tau_2$ if and only if $(w_1, w_2) \in R$ if and only if $\mathbf{M}', \alpha' \models [\![\tau_1]\!] \prec' [\![\tau_2]\!]$ that is $\mathbf{M}', \alpha' \models [\![\tau_1 \prec \tau_2]\!]$.
- $\doteq$-constraints: $\mathbf{M}, \alpha \models \tau_1 \doteq \tau_2$ if and only if $w_1 = w_2$ if and only if $\mathbf{M}', \alpha' \models [\![\tau_1]\!] =_\iota [\![\tau_2]\!]$ that is $\mathbf{M}', \alpha' \models [\![\tau_1 \doteq \tau_2]\!]$.

Step cases: assume that $\mathbf{M}, \alpha \models \varphi @ \tau$ if and only if $\mathbf{M}', \alpha' \models [\![\varphi @ \tau]\!]$, and $\mathbf{M}, \alpha \models \psi @ \tau$ if and only if $\mathbf{M}', \alpha' \models [\![\psi @ \tau]\!]$. Then

- negation: $\mathbf{M}, \alpha \models \neg\varphi @ \tau$ if and only if not $\mathbf{M}, \alpha \models \varphi @ \tau$ if and only if not $\mathbf{M}', \alpha' \models [\![\varphi @ \tau]\!]$ that is $\mathbf{M}', \alpha' \models [\![\neg\varphi @ \tau]\!]$.
- implication: $\mathbf{M}, \alpha \models \varphi \supset \psi @ \tau$ if and only if not $\mathbf{M}, \alpha \models \varphi @ \tau$ or $\mathbf{M}, \alpha \models \psi @ \tau$ if and only if not $\mathbf{M}', \alpha' \models [\![\varphi @ \tau]\!]$ or $\mathbf{M}', \alpha' \models [\![\psi @ \tau]\!]$ that is $\mathbf{M}', \alpha' \models [\![\varphi \supset \psi @ \tau]\!]$.
- quantification: $\mathbf{M}, \alpha \models \forall x.\varphi @ \tau$ if and only if for all $d \in \mathcal{D}$ it is the case that $\mathbf{M}, \alpha^{[d/x]} \models \varphi @ \tau$ if and only if $\mathbf{M}', (\alpha^{[d/x]})' \models [\![\varphi @ \tau]\!]$ if and only if $\mathbf{M}', \alpha'^{[d/x]} \models [\![\varphi @ \tau]\!]$ if and only if $\mathbf{M}', \alpha' \models \forall x.[\![\varphi @ \tau]\!]$ if and only if $\mathbf{M}', \alpha' \models [\![\forall x.\varphi @ \tau]\!]$.
- necessitation ($\Box$): it reduces to the previous cases for quantification and implication, since the domain of quantification of $2FOL$ includes $\mathcal{W}$.

As far as frame properties are concerned, sentences in $\mathsf{FrmAx}(\mathbf{QL})$ enforce exactly those properties of $\prec$ which are needed by the accessibility relation R in order to make the model $\mathbf{M}$ a model of $\mathbf{QL}$ (recall Definition 2 and subsequent discussion). Since $\prec$ is interpreted as R, this completes the proof of Proposition 26. •

Propositions 14, 25 and 26 together lead to

Theorem 27 (Soundness and completeness) *$\mathcal{C}_{\mathbf{QL}}$ is sound and complete for any FO-axiomatisable logic* $\mathbf{QL}$.

5 Conclusions and future work

We have presented a set of labelled sequent calculi for quantified normal modal logics with constant domains and rigid designators whose frame properties can be expressed as a finite set of first-order sentences, possibly employing equality. These calculi are *(i)* modular: each calculus is obtained by adding rules to a basic one; *(ii)* uniform: each added rule clearly enforces a property of the accessibility relation; *(iii)* normalising: frame rules are grouped into a single rule, which is a closing rule; and *(iv)* sound and complete with respect to the corresponding frames. This work extends Viganò's Natural Deduction and sequent systems (see [Viganò, 2000]) to a much wider class of QMLs, although retaining a normalisation property. It is also worth noting that our proof of soundness and completeness is quite different from that given there, and could probably be extended to even more logics.

Future work includes primarily:

1. extension of $\mathcal{C}_{\mathbf{QL}}$ to QMLs with varying domains, to QMLs with flexible designators and to first-order temporal logics;
2. mechanisation of $\mathcal{C}_{\mathbf{QL}}$, extended to first-order linear temporal logic, is in progress[4], and some successes have been obtained so far ([Castellini and Smaill, 2002]).

[4] it is well-known that no complete finitary sequent calculus for full $\mathbf{FOLTL}$ can be given; by an *extension* in this context we mean a calculus obtained by adding rules to $\mathcal{C}_{\mathbf{QS4.3}}$ with no pretence of completeness.

Acknowledgements This work has been carried out at the University of Edinburgh during 1999-2002 and is supported by the EPSRC Grant GR/M46624, "Mechanising first-order temporal logics". The authors wish to thank Silvio Ghilardi, Rajeev Goré and Luca Viganò for several insightful private communications.

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