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Institute of Perception, Action and Behaviour

Euclidean Fitting Revisited

by

Petko Faber, Robert Fisher

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Abstract :

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Abstract

The focus of our paper is on the fitting of general curves and surfaces to 3D data. In the past researchers have used approximate distance functions rather than the Euclidean distance because of computational efficiency. We now feel that machine speeds are sufficient to ask whether it is worth considering Euclidean fitting again. Experiments with the real Euclidean distance show the limitations of suggested approximations like the Algebraic distance or Taubin's approximation. In this paper we present our results improving the known fitting methods by an (iterative) estimation of the real Euclidean distance. The performance of our method is compared with several methods proposed in the literature and we show that the Euclidean fitting guarantees a better accuracy with an acceptable computational cost.

1 Motivation

One fundamental problem in building a recognition and positioning system based on implicit 3D curves and surfaces is how to fit these curves and surfaces to 3D data. This process will be necessary for automatically constructing CAD or other object models from range or intensity data and for building intermediate representations from observations during recognition. Of great importance is the ability to represent 2D and 3D data or objects in a compact form. Implicit polynomial curves and surfaces are very useful representations. Their power appears by their ability to smooth noisy data, to interpolate through sparse or missing data, their compactness and their form being commonly used in numerous constructions. Let $f_2(\vec{x})$ be an *implicit polynomial* of degree 2 given by

$$f_2(\vec{x}) = p_0 + \vec{x}' \cdot \vec{p}_1 + \vec{x}' \cdot \vec{P}_2 \cdot \vec{x} = 0, \ \vec{x} \in \mathbb{R}^2 \text{ or } \vec{x} \in \mathbb{R}^3 \ . \tag{1}$$

Then, we only have to determine the set of parameters which describes the data best. The parameter estimation problem is usually formulated as an optimization problem. Thereby, a given estimation problem can be solved in many ways because of different optimization criteria and several possible parameterizations. Generally, the literature on fitting can be divided into two general techniques: clustering (e.g. [4, 6]) and least-squares fitting (e.g. [2, 5, 7]). While the clustering methods are based on mapping data points to the parameter space, such as the Hough transform and the accumulation methods, the least-squares methods are centered on finding the sets of parameters that minimize some distance measures between the data points and the curve or surface. Unfortunately, the minimization of the Euclidean distances from the data points to a *general* curve or surface has been computationally impractical, because there is no closed form expression for the Euclidean distance from a point to a general algebraic curve or surface, and iterative methods are required to compute it. Thus, the Euclidean distance has been approximated. Often, the result of evaluating the characteristic polynomial $f_2(\vec{x})$ is taken, or the first order approximation, suggested by Taubin [12] is used. However, experiments with the Euclidean distance show the limitations of approximations regarding quality and accuracy of the fitting results.

The quality of the fitting results has a substantive impact on the recognition performance especially in the reverse engineering where we work with a constrained reconstruction of 3D geometric models of objects from range data. Thus it is important to get good fits to the data.

2 Fitting of algebraic curves and surfaces

An implicit curve or surface is the set of zeros of a smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$ of the n variables: $\mathcal{Z}(f) = \{\vec{x} : f(\vec{x}) = 0\}$. In our applications we are interested in three special cases for their applications

in computer vision and especially range image analysis: $\mathcal{Z}(f)$ is a *planar curve* if $n = 2$ and $k = 1$, it is a *surface* if $n = 3$ and $k = 1$ and it is a *space curve* if $n = 3$ and $k = 2$.

Given a finite set of data points $\mathcal{D} = \{\vec{x}_i\}$, $i \in [1, m]$, the problem of fitting an algebraic curve or surface $\mathcal{Z}(f)$ to the data set $\mathcal{D}$ is usually cast as minimizing the mean square distance

$$\frac{1}{m}\sum_{i=1}^{m} \mathrm{dist}\,(\vec{x}_i, \mathcal{Z}(f))^2 \rightarrow \text{ Minimum} \tag{2}$$

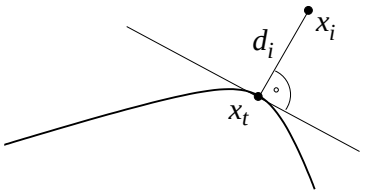

Figure 1: Euclidean distance $\mathrm{dist}(\vec{x}_i, \mathcal{Z}(f))$ of a point $\vec{x}_i$ to a zero set $\mathcal{Z}(f)$

from the data points to the curve or surface $\mathcal{Z}(f)$, a function of the set of parameters of the polynomial. The problem that we have to deal with is how to answer whether the distance from a certain point $\vec{x}_i$ to a set $\mathcal{Z}(f)$ of zeros of $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is the (global) minimum or not. The distance from the point $\vec{x}_i$ to the zero set $\mathcal{Z}(f)$ is defined as the minimum of the distances from $\vec{x}_i$ to points $\vec{x}_t$ in the zero set $\mathcal{Z}(f)$

$$\mathrm{dist}(\vec{x}_i, \mathcal{Z}(f)) = \min\{\| \vec{x}_i - \vec{x}_t \| : f(\vec{x}_t) = 0\}. \tag{3}$$

Thus, the Euclidean distance $\mathrm{dist}(\vec{x}_i, \mathcal{Z}(f))$ between a point $\vec{x}_i$ and the zero set $\mathcal{Z}(f)$ is the minimal distance between $\vec{x}_i$ and the point $\vec{x}_t$ in the zero set whose tangent is orthogonal to the line joining $\vec{x}_i$ and $\vec{x}_t$ (see Fig.1). As mentioned above there is no closed form expression for the Euclidean distance from a point to a *general* algebraic curve or surface and iterative methods are required to compute it. In the past researchers have often replaced the Euclidean distance by an approximation. But it is well known that a different performance function can produce a very biased result. In the following we will summarize the methods used to approximate the real Euclidean distance by the algebraic distance and an approximation suggested by Taubin ([12], [13]).

Algebraic fitting. The algebraic fitting is based on the approximation of the Euclidean distance between a point and the curve or surface by the algebraic distance

$$\mathrm{dist}_A\,(\vec{x}_i, \mathcal{Z}(f)) = f_2(\vec{x}_i)\ . \tag{4}$$

To avoid the trivial solution, where all parameters are zero, and any multiple of a solution, the parameter vector may be constrained in some way (e.g. [1, 5, 7] and [10]). The pros and cons of using algebraic distances are a) the gain in computational efficiency, because closed form solutions can usually be obtained, on the one hand and b) the often unsatisfactory results on the other hand.

Taubin's fitting. An alternative to approximately solve the minimization problem is to replace the Euclidean distance from a point to an implicit curve or surface by the first order approximation [13]. There, the Taylor series is expanded up to first order in a defined neighborhood, truncated after the linear term and then the triangular and the Cauchy-Schwartz inequality were applied.

$$\mathrm{dist}_T\,(\vec{x}_i, \mathcal{Z}(f)) = \frac{|f_2(\vec{x}_i)|}{\|\nabla f_2(\vec{x}_i)\|} \tag{5}$$

Besides the fact that no iterative procedures are required, the fundamental property is that it is a first order approximation to the exact distance. But, it is important to note that the approximate distance is also biased in some sense. If, for instance, a data point $\vec{x}_i$ is close to a critical point of the polynomial, i.e., $\|\nabla f_2(\vec{x}_i)\| \approx 0$, but $f_2(\vec{x}_i) \neq 0$, the distance becomes large. This is certainly a limitation.

Note, neither the Algebraic distance nor Taubin's approximation are invariant with respect to Euclidean transformations.

2.1 Euclidean distance

To overcome the problems with the approximated distances, it is natural to replace them again by the real geometric distances, that means the Euclidean distances, which are invariant to transformations in Euclidean space and are not biased. For primitive curves and surfaces like straight lines, ellipses, planes, cylinders, cones, and ellipsoids, a closed form expression exists for the Euclidean distance from a point to the zero set and we use these. However, as the expression of the Euclidean distance to other 2nd order curve and surfaces is more complicated and there exists no known closed form expression, an iterative optimization procedure must be carried out. For more general curves and surfaces the following simple iterative algorithm will be used (see also Fig.2):

1. *Select* the initial point $\vec{x}_t^{[0]}$. In the first step we determine the initial solution by intersecting the curve or surface with the straight line defined by the center point $\vec{x}_m$ and the point $\vec{x}_i$. By the initial solution, the upper bound for the distance is estimated.

2. *Update* the actual estimation $\vec{x}_t^{[k+1]} = F(\vec{x}_t^{[k]})$, $k = 0, 1, 2, \ldots$. In the second step a new solution is determined. The search direction will be determined by the gradient of the curve $\nabla(f(\vec{x}_t^{[k]}))$.

$$\vec{x}_t^{[k+1]} = \vec{x}_t^{[k]} + \alpha^{[k]} \nabla f \left(\vec{x}_t^{[k]} \right). \tag{6}$$

   The method is an adaptation of the steepest descent method. As the result we get two possible solutions, $\vec{x}_t^{[k]}$ and $\vec{x}_t^{[k+1]}$ (cf. Fig 2), and we have to decide by an objective function $\mathcal{F}$, if $\vec{x}_t^{[k+1]}$ will be accepted as new solution.

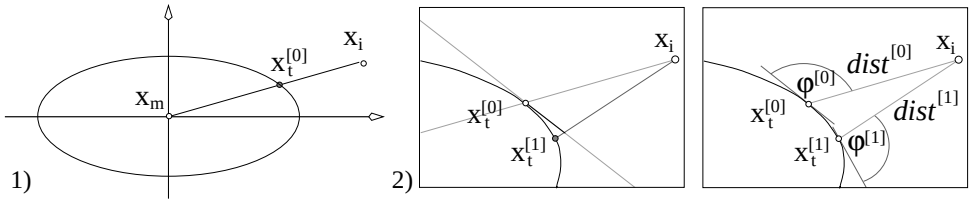

Figure 2: Steps to estimate the Euclidean distance $\text{dist}_E(\vec{x}_i, \mathcal{Z}(f))$ of a point $\vec{x}_i$ to the zero set $\mathcal{Z}(f)$ of an ellipse

3. *Evaluate* the new estimation $\vec{x}_t^{[k+1]}$. The set of solutions is evaluated by the objective function $\mathcal{F}(\vec{x}_i, \vec{x}_t^{[k+1]}, \mathcal{Z}(f)) = \min(\text{dist}_E(\vec{x}_i, \vec{x}_t^{[k]}), \text{dist}_E(\vec{x}_i, \vec{x}_t^{[k+1]})$. If the distance from the new estimation $\vec{x}_t^{[k+1]}$ is smaller, we accept this as the new local solution. Otherwise $\vec{x}_t^{[k+1]} = \vec{x}_t^{[k]}$ and $\alpha^{[k+1]} = -\tau\alpha^{[k]}$, $\tau > 0$. Then, the algorithm will be continued with step 2 until the difference between the distances of the old and the new estimation is smaller then a given threshold. To speed up the estimation a criterion to terminate the updating may be used like e.g. $\|\vec{x}_t^{[k+1]} - \vec{x}_t^{[k]}\| \leq \tau_d$, or $k \geq \tau_k$.

2.2 Estimation error of surface fit

Given the Euclidean distance error for each point, we then compute the curve or surface fitting error as $\text{dist}_E(\vec{x}_i, \mathcal{Z}(f))$. The standard least-squares method tries to minimize $\sum_i \text{dist}_E^2(\vec{x}_i, \mathcal{Z}(f))$, which is unstable if there are outliers in the data. Outlying data can give so strong an effect in the minimizing that the parameters are distorted. Replacing the squared residuals by another function can reduce the effect of outliers. Appropriate minimization criteria including functions were discussed in for instance [3] and [14]. It seems difficult to select a function which is generally suitable. Following the results given in [11] the best choice may be the so-called L_p (*least power*) function: $L_p := |\text{dist}_E(\vec{x}_i, \mathcal{Z}(f))|^\nu / \nu$. This function represents a family of functions including the two commonly used functions L_1 (*absolute power*) with $\nu = 1$ and L_2 (*least squares*) with $\nu = 2$. Note, the smaller ν, the smaller is the influence of large errors. For values $\nu \approx 1.2$, a good error estimation may be expected [11].

2.3 Optimization

Given a method of computing the fitting error for the curves and surfaces, we now show how to minimize the error. Many techniques are readily available, including Gauss-Newton algorithm, Steepest Gradient Descent, and Levenberg-Marquardt algorithm. Our implementation is based on the Levenberg-Marquardt (LM) algorithm [8, 9] which has become the standard of nonlinear optimization routines. The LM method combines the inherent stability of the Steepest Gradient Descent with the quadratic convergence rate of the Gauss-Newton method. The (iterative) fitting approach consists of three major steps:

1. *Select* the initial fitting $\mathcal{P}^{[0]}$. The initial solution $\mathcal{P}^{[0]}$ is determined by Taubin's fitting method.

2. *Update* the estimation $\mathcal{P}^{[k+1]} = F_{\text{LM}}(\mathcal{P}^{[k]})$ using the Levenberg-Marquardt (LM) algorithm.

3. *Evaluate* the new estimation $\mathcal{P}^{[k+1]}$. The updated parameter vector is evaluated using the L_p function on the basis of the $\text{dist}_E(\vec{x}_i, \mathcal{Z}(f))$. $\mathcal{P}^{[k+1]}$ will be accepted if $L_p(\mathcal{P}^{[k+1]}) < L_p(\mathcal{P}^{[k]})$ and the fitting will be continued with step 2. Otherwise the fitting is terminated and $\mathcal{P}^{[k]}$ is the desired solution.

3 Experimental results

We present experimental results comparing Euclidean fitting (EF) with Algebraic fitting (AF), and Taubin's fitting (TF) in terms of quality, robustness and speed.

3.1 Robustness

To test the robustness of the proposed EF method, we used three different surface types: cylinders, cones, and general quadrics. Note that plane estimation is the same for all three methods. To enforce the fitting of a special surface type we include in all three fitting methods the same constraints which describe the expected surface type. The 3D data were generated by adding isotropic Gaussian noise $\sigma = \{1\%, 5\%, 10\%, 20\%\}$. Additionally the surfaces were partially occluded. The visible surfaces were varied between 1/2 (maximal case), 5/12, 1/3, 1/4, and 1/6 of the full 3D cylinder (see Fig.4). In all our

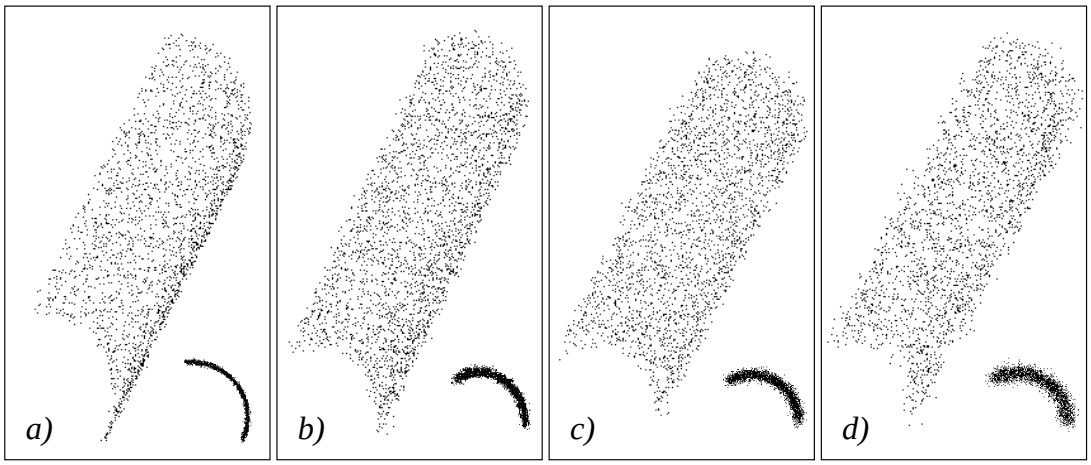

Figure 3: View of the 3D data points for a cylinder (maximal case) with added isotropic Gaussian noise a) $\sigma = 1\%$, b) $\sigma = 5\%$, c) $\sigma = 10\%$, and d) $\sigma = 20\%$.

experiments the number of 3D points was 5000. And finally, each experiment runs 100 times to measure the average fitting error. The mean least power errors ($MLPE$'s) of the different fittings are in Tab.1. We determined the real geometric distance between the 3D data points and the estimated surfaces using the method described in Sec.2.1. That means we calculated the $MLPE$ for all fitting results on the basis of the estimated Euclidean distance. Otherwise, a comparison of the results will be useless. Based on this table we evaluate the three fitting methods with respect to quality and robustness. The EF requires

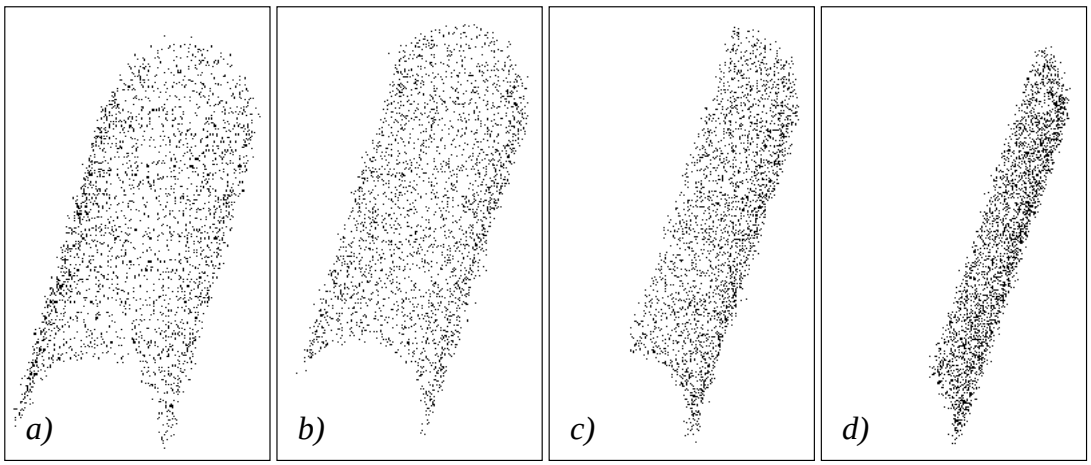

Figure 4: View of the 3D data points of partially occluded cylinders, a) 1/2 (maximal case), b) 5/12, 1/3 (see Fig.3b)), c) 1/4, and d) 1/6. Added isotropic Gaussian noise $\sigma = 5\%$.

an initial estimate for the parameters, and we have found that the results depend on the initial choice. A quick review of the values in Tab.1 shows that the results of TF are better for initializing than the

results of *AF*. Maybe another fitting method can give a better initialization, but here we use *TF* because of its advantages. As expected, the *TF* and *EF* yield the best results respect with to the mean and

			AF	*TF*	*EF*
			$[\ \mu \quad \sigma\ ] \cdot 10^{-2}$	$[\ \mu \quad \sigma\ ] \cdot 10^{-2}$	$[\ \mu \quad \sigma\ ] \cdot 10^{-2}$
cylinder_1 (rad=50, length=500)	$\sigma = 1\%$	6/12	$[\ 9.06 \pm\ 1.55]_{(0.08)}$	$[\ 1.14 \pm\ 0.06]$	$[\ 0.71 \pm\ 0.03]$
		5/12	$[33.19 \pm\ 3.90]_{(0.22)}$	$[\ 1.30 \pm\ 0.13]$	$[\ 0.65 \pm\ 0.04]$
		4/12	$[44.91 \pm\ 3.72]_{(0.02)}$	$[\ 2.75 \pm\ 0.43]$	$[\ 0.72 \pm\ 0.05]$
		3/12	$[55.06 \pm\ 2.04]_{(0.08)}$	$[\ 3.82 \pm\ 0.80]$	$[\ 0.94 \pm\ 0.11]$
		2/12	$[58.05 \pm\ 3.07]_{(0.13)}$	$[\ 3.94 \pm\ 0.49]$	$[\ 1.30 \pm\ 0.10]$
	$\sigma = 5\%$	6/12	$[14.80 \pm\ 1.88]_{(0.03)}$	$[\ 1.32 \pm\ 0.17]$	$[\ 0.62 \pm\ 0.03]$
		5/12	$[36.11 \pm\ 2.21]_{(0.14)}$	$[\ 2.92 \pm\ 0.13]$	$[\ 0.93 \pm\ 0.12]$
		4/12	$[25.56 \pm\ 2.76]_{(0.08)}$	$[\ 5.27 \pm\ 0.32]$	$[\ 1.35 \pm\ 0.32]$
		3/12	$[55.13 \pm\ 3.50]_{(0.02)}$	$[\ 5.07 \pm\ 0.75]$	$[\ 1.97 \pm\ 0.45]$
		2/12	$[55.93 \pm\ 3.08]_{(0.15)}$	$[\ 4.13 \pm\ 1.03]$	$[\ 2.34 \pm\ 0.81]$
	$\sigma = 10\%$	6/12	$[10.44 \pm\ 1.09]_{(0.10)}$	$[\ 2.05 \pm\ 0.29]$	$[\ 0.97 \pm\ 0.12]$
		5/12	$[23.10 \pm\ 3.47]_{(0.18)}$	$[\ 3.48 \pm\ 0.74]$	$[\ 1.71 \pm\ 0.63]$
		4/12	$[37.76 \pm\ 3.45]_{(0.23)}$	$[\ 3.90 \pm\ 0.79]$	$[\ 1.78 \pm\ 0.49]$
		3/12	$[56.37 \pm\ 2.73]_{(0.09)}$	$[\ 4.28 \pm\ 1.04]$	$[\ 1.83 \pm\ 0.34]$
		2/12	$[58.46 \pm\ 3.33]_{(0.03)}$	$[\ 8.98 \pm\ 3.46]$	$[\ 3.02 \pm\ 1.13]$
	$\sigma = 20\%$	6/12	$[15.34 \pm\ 1.93]_{(0.03)}$	$[\ 2.38 \pm\ 0.35]$	$[\ 1.09 \pm\ 0.10]$
		5/12	$[49.40 \pm\ 3.18]_{(0.13)}$	$[\ 2.90 \pm\ 0.56]$	$[\ 1.07 \pm\ 0.07]$
		4/12	$[55.61 \pm\ 3.24]_{(0.11)}$	$[\ 3.69 \pm\ 0.64]$	$[\ 1.41 \pm\ 0.11]$
		3/12	$[22.49 \pm\ 2.66]_{(0.10)}$	$[\ 4.05 \pm\ 0.95]$	$[\ 1.72 \pm\ 0.38]$
		2/12	$[41.37 \pm\ 3.22]_{(0.12)}$	$[\ 9.20 \pm\ 3.55]$	$[\ 3.00 \pm\ 1.13]$
cylinder_2 (rad=250, length=500)	$\sigma = 1\%$	6/12	$[26.68 \pm\ 1.37]_{(0.02)}$	$[\ 6.40 \pm\ 0.05]$	$[\ 4.26 \pm\ 0.02]$
		5/12	$[21.82 \pm\ 0.96]$	$[\ 6.83 \pm\ 0.13]$	$[\ 3.68 \pm\ 0.21]$
		4/12	$[25.39 \pm\ 0.88]$	$[\ 7.28 \pm\ 0.25]$	$[\ 4.12 \pm\ 0.17]$
		3/12	$[21.93 \pm\ 1.76]_{(0.01)}$	$[10.67 \pm\ 0.74]$	$[\ 3.18 \pm\ 0.48]$
		2/12	$[25.80 \pm\ 2.26]$	$[23.24 \pm\ 1.67]$	$[\ 5.43 \pm\ 0.70]$
	$\sigma = 5\%$	6/12	$[25.31 \pm\ 1.26]_{(0.03)}$	$[\ 6.67 \pm\ 0.21]$	$[\ 4.73 \pm\ 0.33]$
		5/12	$[18.54 \pm\ 0.83]$	$[\ 8.06 \pm\ 0.71]$	$[\ 3.22 \pm\ 0.18]$
		4/12	$[25.32 \pm\ 1.05]$	$[\ 8.38 \pm\ 0.72]$	$[\ 5.26 \pm\ 0.70]$
		3/12	$[18.29 \pm\ 1.10]_{(0.07)}$	$[15.91 \pm\ 1.60]$	$[\ 6.47 \pm\ 1.08]$
		2/12	$[40.08 \pm\ 1.90]_{(0.02)}$	$[25.38 \pm\ 1.57]$	$[\ 8.88 \pm\ 1.36]$
	$\sigma = 10\%$	6/12	$[26.27 \pm\ 1.45]_{(0.09)}$	$[\ 6.71 \pm\ 0.23]$	$[\ 3.99 \pm\ 0.30]$
		5/12	$[19.31 \pm\ 0.84]$	$[\ 7.46 \pm\ 0.48]$	$[\ 3.79 \pm\ 0.45]$
		4/12	$[27.33 \pm\ 0.84]$	$[\ 8.19 \pm\ 0.90]$	$[\ 4.11 \pm\ 0.52]$
		3/12	$[23.42 \pm\ 1.92]$	$[15.87 \pm\ 1.70]$	$[\ 5.32 \pm\ 0.85]$
		2/12	$[31.89 \pm\ 1.85]_{(0.02)}$	$[25.68 \pm\ 2.04]$	$[\ 7.15 \pm\ 0.92]$
	$\sigma = 20\%$	6/12	$[24.74 \pm\ 1.33]_{(0.02)}$	$[\ 6.80 \pm\ 0.27]$	$[\ 3.49 \pm\ 0.13]$
		5/12	$[18.55 \pm\ 0.87]_{(0.07)}$	$[\ 7.11 \pm\ 0.56]$	$[\ 3.95 \pm\ 0.33]$
		4/12	$[27.08 \pm\ 0.90]$	$[\ 7.43 \pm\ 0.35]$	$[\ 5.24 \pm\ 0.35]$
		3/12	$[22.32 \pm\ 1.36]_{(0.04)}$	$[15.17 \pm\ 1.79]$	$[\ 6.60 \pm\ 0.91]$
		2/12	$[35.18 \pm\ 2.28]_{(0.02)}$	$[38.71 \pm\ 8.81]$	$[11.30 \pm\ 2.22]$

Table 1: Least power error fitting cylinder 1 and 2. The visible surfaces were varied between 1/2 (maximal case), 5/12, 1/3, 1/4, and 1/6 of the full 3D cylinder. Gaussian noise σ was 1%, 5%, 10%, and 20%. For *AF* the percentage of failed fittings is given in brackets. The number of trials was 100.

standard deviation, and the mean for *EF* is always lower than for the other two algorithms. The results of *AF* are not acceptable because of their high values for mean and standard deviation. The results of *TF* are much better, compared with the *AF*. But, in the direct comparison with the *EF* these results are also unacceptable. Furthermore, note that *AF* give sometimes wrong results which means that the fitted curve or surface typo does not come up with our expectations. We removed all failed fittings out of the considerations. The percentage of failures is given as footnote in Tab.1. For *TF* and *EF* we had no failures in our experiments.

3.2 Noise sensitivity

The second experiment is perhaps more important and assesses the stability of the fitting with respect to different realizations of noise with the same variance. The noise has been set to a relatively high level because the limits of the three methods are more visible then. It is very desirable that the performance is affected only by the noise level, and not by a particular realization of the noise. In Tab.1 the μ's and σ's are shown for four different noise levels. If we analyze the table regarding noise sensitivity, we observe:

- The stability of all fittings, reflected in the standard deviation, is influenced by the noise level of the data. The degree of occlusion has an additional influence on stability. Particularly serious is the combination of both high noise level($\sigma \geq 20\%$) and strong occlusion (visible surface $< 1/4$).
- *AF* is very unstable, even with a noise level of $\sigma = 1\%$. In some experiments with *AF* the fitting failed and the estimated *mean least power error* between the estimated surface and the real 3D data was greater than a given threshold. We removed all failed fittings, sometimes up to 23 percent (see Tab.1: fitting cylinder 1, 1/4 visible and $\sigma = 10\%$). Thus, the performance of the Algebraic fitting is strongly affected by the particular realization of the noise, which is absolutely undesirable.
- *TF* is also affected by particular instances of the noise, but on a significantly lower level.
- The noise sensitivity of *EF* has a similar good performance. The cause for the instability of the *EF* is the initialization.

3.3 Sample density

In the third experiment we examined the influence of the sample density. The cardinality of the 3D point set was varied accordingly. On the basis of the *MLPE* for the several fittings (see Tab.2) it can be seen that, with increasing the number of points, the fitting becomes a) more robust and b) less noise sensitive. Note, not only is the absolute number of points important, but the point density is crucial.

However noise sensitivity increases with increasing occlusion for both *TF* and *EF*, so that the fitting becomes altogether more unstable. Similar conclusions about *AF* as in Sec.3.1 and Sec.3.2 also apply here.

3.4 Computational cost

The algorithms have been implemented in C and the computation was performed on a SUN Sparc ULTRA 5 workstation. The average computational costs in milliseconds per 1000 points for the three algorithms are in Tab.3.

As expected. the *AF* and *TF* supply the best performance, because the *EF* algorithm requires a repeated search for the point x_t closest to x_i and the calculation of the Euclidean distance. A quick review of the values in Tab.3 shows that the computational costs increase if we fit an elliptical cylinder, a circular or an elliptical cone respectively a general quadric. The algorithm to estimate the distance by the closed form solution respectively the iterative algorithm is more complicated in these cases (cf. Sec.2.1).

The number of necessary iterations is also influenced by the required precision of the LM algorithm to terminate the updating process.

4 Conclusion

We revisited the Euclidean fitting of curves and surfaces to 3D data to investigate if it is worth considering Euclidean fitting again. The focus was on the quality and robustness of Euclidean fitting compared with the commonly used Algebraic fitting and Taubin's fitting. Now, we can conclude that robustness and accuracy increases sufficiently compared to both other methods and Euclidean fitting is more stable with increased noise.

The main disadvantage of the Euclidean fitting, computational cost, has become less important due to rising computing speed. In our experiments the computational costs of Euclidean fitting were only about 2-19 times worse than Taubin's fitting. This relation probably cannot be improved substantially in favor of Euclidean fitting, but the absolute computational costs are becoming an insignificant deterrent to usage, especially if high accuracy is required.

			AF	*TF*	*EF*
			$[\ \mu \quad \sigma\ ] \cdot 10^{-2}$	$[\ \mu \quad \sigma\ ] \cdot 10^{-2}$	$[\ \mu \quad \sigma\ ] \cdot 10^{-2}$
cylinder_1	500	6/12	$[15.88 \pm 2.54]_{(0.03)}$	$[2.94 \pm 0.80]$	$[1.17 \pm 0.30]$
		5/12	$[29.73 \pm 3.31]_{(0.03)}$	$[1.55 \pm 0.11]$	$[0.86 \pm 0.06]$
		4/12	$[32.96 \pm 2.55]_{(0.05)}$	$[4.39 \pm 0.90]$	$[2.30 \pm 0.68]$
		3/12	$[23.67 \pm 3.01]_{(0.04)}$	$[3.81 \pm 0.51]$	$[1.55 \pm 0.12]$
		2/12	$[24.36 \pm 1.51]_{(0.06)}$	$[6.86 \pm 2.45]$	$[4.37 \pm 1.63]$
	1000	6/12	$[16.61 \pm 2.42]_{(0.08)}$	$[1.57 \pm 0.17]$	$[0.85 \pm 0.11]$
		5/12	$[36.17 \pm 3.52]_{(0.17)}$	$[3.52 \pm 1.40]$	$[1.75 \pm 0.59]$
		4/12	$[35.06 \pm 2.70]_{(0.06)}$	$[2.79 \pm 0.33]$	$[1.24 \pm 0.15]$
		3/12	$[22.01 \pm 3.03]_{(0.02)}$	$[4.51 \pm 0.71]$	$[1.68 \pm 0.16]$
		2/12	$[25.43 \pm 1.91]$	$[4.10 \pm 1.17]$	$[1.47 \pm 0.12]$
	2000	6/12	$[16.85 \pm 3.00]_{(0.08)}$	$[1.61 \pm 0.26]$	$[0.72 \pm 0.03]$
		5/12	$[45.99 \pm 2.94]_{(0.07)}$	$[1.93 \pm 0.43]$	$[0.73 \pm 0.04]$
		4/12	$[38.38 \pm 2.81]_{(0.15)}$	$[3.30 \pm 0.76]$	$[1.22 \pm 0.29]$
		3/12	$[19.79 \pm 2.52]_{(0.06)}$	$[4.60 \pm 1.16]$	$[2.05 \pm 0.56]$
		2/12	$[24.28 \pm 1.64]_{(0.07)}$	$[2.22 \pm 0.28]$	$[1.30 \pm 0.08]$
cylinder_2	500	6/12	$[23.27 \pm 0.98]$	$[7.08 \pm 0.24]$	$[4.10 \pm 0.32]$
		5/12	$[20.05 \pm 1.94]$	$[7.82 \pm 0.56]$	$[3.72 \pm 0.22]$
		4/12	$[25.19 \pm 1.30]$	$[10.55 \pm 0.62]$	$[4.66 \pm 0.28]$
		3/12	$[21.12 \pm 1.85]_{(0.04)}$	$[17.10 \pm 1.68]$	$[7.18 \pm 0.87]$
		2/12	$[38.13 \pm 2.77]_{(0.17)}$	$[24.74 \pm 2.86]$	$[7.90 \pm 0.81]$
	1000	6/12	$[25.84 \pm 1.28]_{(0.05)}$	$[7.11 \pm 0.31]$	$[3.97 \pm 0.39]$
		5/12	$[20.19 \pm 1.59]$	$[7.60 \pm 0.37]$	$[3.33 \pm 0.19]$
		4/12	$[25.11 \pm 0.89]$	$[8.37 \pm 0.40]$	$[4.23 \pm 0.17]$
		3/12	$[20.61 \pm 1.33]_{(0.06)}$	$[14.82 \pm 1.21]$	$[8.27 \pm 1.03]$
		2/12	$[38.37 \pm 2.73]_{(0.10)}$	$[21.61 \pm 1.76]$	$[6.66 \pm 0.84]$
	2000	6/12	$[24.30 \pm 1.11]_{(0.02)}$	$[6.39 \pm 0.17]$	$[3.74 \pm 0.29]$
		5/12	$[18.95 \pm 0.82]_{(0.02)}$	$[7.03 \pm 0.32]$	$[3.08 \pm 0.17]$
		4/12	$[27.31 \pm 0.87]$	$[7.98 \pm 0.34]$	$[4.36 \pm 0.16]$
		3/12	$[20.31 \pm 1.02]_{(0.06)}$	$[14.92 \pm 1.50]$	$[5.36 \pm 0.79]$
		2/12	$[35.16 \pm 2.15]_{(0.05)}$	$[24.90 \pm 1.76]$	$[8.82 \pm 1.21]$

Table 2: Mean squares error for cylinder fitting by varied sample density. The density was 500, 1000, and 2000 3D points. Gaussian noise was $\sigma = 5\%$. For AF the percentage of failed fittings is given as a footnote.

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	Average computational costs [msec.]		
	AF	*TF*	*EF*
Plane	0.958	1.042	2.417
Sphere	1.208	1.250	3.208
Circular cylinder	3.583	3.625	12.375
Elliptical cylinder	13.292	13.958	241.667
Circular cone	15.667	15.833	288.375
Elliptical cone	15.042	15.375	291.958
General quadric	18.208	18.458	351.083

Table 3: Average computational costs in milliseconds per 1000 points.

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